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Latitude Variation of 27-Day Cosmic Ray Intensity Decreases (*).

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Summary. — The latitude variation of cosmic ray intensity decreases due to the modulation effects of a geocentric nebula of disordered magnetic fields has been calculated. Experimental data provided by neutron monitor observations during a period of intense solar activity, when compared with the results of these calculations, show that equatorial variations exceed those given by the modulation mechanism by a factor of two or more.

1. – Introduction.

As a result of investigations of the intensity (1) and latitude variation (2-4) of cosmic radiation over the 11-year solar cycle, it has become evident that cosmic radiation displays modulation effects resulting from solar-controlled mechanisms. In an attempt to account for these variations, PARKER (5) has proposed that the modulation is due to changes in the spectra of cosmic radiation as a result of diffusion of primary particles through a geocentric nebula of disordered magnetic fields. The changes in spectra produced in this way are confined largely to the latitude sensitive region and depend on the field

(*) Supported in part by the joint program of the U.S. Office of Naval Research and the U.S. Atomic Energy Commission. Reproduction in whole or in part is permitted for any purpose of the United States Government.

(1) S. E. FORBUSH: *Journ. Geophys. Res.*, **59**, 525 (1954).

(2) H. V. NEHER: *Phys. Rev.*, **100**, 959 (1955).

(3) P. MEYER and J. A. SIMPSON: *Phys. Rev.*, **99**, 1517 (1955).

(4) P. MEYER and J. A. SIMPSON: *Phys. Rev.*, **106**, 568 (1957).

(5) E. N. PARKER: *Phys. Rev.*, **103**, 1518 (1956).

strength and number density of the magnetized gas clouds in the vicinity of the earth.

A direct test of this mechanism would entail investigating the primary spectra of cosmic radiation over widely different levels of solar activity; for example, comparing the spectra at solar minimum with those of solar maximum, or determining the changes in the spectra during periods of enhanced solar activity. A less direct test, nevertheless indicative of the role of this mechanism in describing these phenomena, would be provided by measurements of the latitude effect, over a range of latitudes extending from the geomagnetic equator to above the knee of the latitude curve, under similar conditions. In principle, these measurements could be obtained by rapidly transporting a single detector over a wide range of latitudes or from a number of detectors distributed over the same range. In practice, a net-work of detectors provides the simplest means of studying these, as well as other, effects.

It is the purpose of this paper to compare experimental data provided by neutron monitor observations during a period of intense solar activity with the results of calculations on the latitude variation of the fractional change in intensity for neutron monitors, based on the modulation mechanism proposed by PARKER. From this comparison it becomes evident that features of the mechanism proposed by PARKER which extend modulation effects to relatively high rigidities must be considered to bring the results of calculations with this model into agreement with these observations.

2. – Observations.

The first outbreak of intense solar activity of the current solar cycle developed in mid-January 1956, and was accompanied by numerous solar flares, radio noise outbursts, ionospheric disturbances, and geomagnetic storms, in addition to cosmic ray intensity variations. In considering the latitude variation of cosmic ray modulation effects, data on the intensity decreases during the months of January and February, 1956, have been assembled from observations provided by mountain altitude neutron monitors distributed over a wide range in geomagnetic latitudes. These data show that the intensity variations were world-wide decreases, approximately 12 to 14 days in duration, and reached minimum intensity on about January 21 and February 20 respectively.

The amplitudes of the intensity decreases observed in February 1956 with four neutron monitors ([6-9]), expressed relative to the intensity just prior to

([6]) P. MEYER, E. N. PARKER and J. A. SIMPSON: *Phys. Rev.*, **104**, 768 (1956).

([7]) J. A. LOCKWOOD: private communication.

([8]) R. R. BROWN: *Journ. Geophys. Res.*, **61**, 639 (1956).

([9]) M. WADA: private communication.

the outbreak of solar activity in mid-January, 1956, are given in Table I, together with the atmospheric depths, geomagnetic latitudes and proton cut-off momenta of the stations as given by PFOTZER [10] using the westward shifted eccentric dipole proposed by SIMPSON *et al.* [11].

The amplitude of the decrease given in Table I for Mt. Norikura, Japan, represents a lower limit for the neutron variations at this latitude. This results from the fact that data were not available for the day of minimum intensity in February and data from three days earlier were used instead. The actual intensity decrease at Mt. Norikura was probably about 1% larger that that given in Table I.

TABLE I.

Station	Atm. depth (g/cm²)	Latitude (eff. system)	Proton cut-off momenta (GeV/c)	Intensity decrease
Climax, Colo.	645	50.3°	2.70	10 %
Mt. Washington, N. H. .	820	49.7°	3.05	9.1%
Albuquerque, N.M.	860	47°	3.50	7.5%
Mt. Norikura, Japan . . .	725	31.6°	8.4 (*)	5.4%
			10.7 (+)	
Huancayo, Peru	645	— 3.6°	13.8	(2.4%) (×)

(*) Value related to the Stoermer Cone.
(+) Value related to the Main Cone.
(×) Ion Chamber Data.

Also included in Table I is the amplitude of the intensity decrease observed at Huancayo, Peru, with an ion chamber [12]. Since intensity variations observed with neutron monitors are larger than those obtained with instruments sensitive to the μ-meson component at the same latitude, the ion chamber data represent an extreme lower limit for the neutron variations near the geomagnetic equator. As a rough estimate of the neutron variations at Huancayo, the ion chamber amplitude should be increased by a factor of two or so.

3. – Calculations.

3·1. *General.* – The calculation of the latitude variation of the fractional change in neutron intensity during periods of strong solar activity was based

(10) G. PFOTZER: *Proceedings of the* 1957 *IUPAP Cosmic Ray Conference*, in *Suppl. Nuovo Cimento* (in press).

(11) J. A. SIMPSON, K. B. FENTON, J. KATZMAN and D. C. ROSE: *Phys. Rev.*, **102**, 1648 (1956).

(12) S. E. FORBUSH: private communication.

on the methods outlined by TREIMAN *et al.* ([13]), FONGER ([14]), and SIMPSON ([15]). Thus the neutron intensity due to vertically incident primary particles I_v is given by

$$I_v(\lambda, x, t) = \sum_Z k_Z \int_{E_Z(\lambda,t)}^{\infty} j_Z(E, t)\, S(E, x)\, \mathrm{d}E\,,$$

where $j_Z(E, t)$ represents the differential energy spectrum for particles of atomic number Z, k_Z is a weighting factor proportional to the number of nucleons per incident primary particle, $S(E, x)$ is the yield function which gives the rate of local neutron production at atmospheric depth x resulting from unit flux of vertically incident nucleons of energy E and $E_Z(\lambda, t)$ is the cut-off energy per nucleon for nuclei with charge Z at geomagnetic latitude λ. For large atmospheric depths, the neutron intensity resulting from vertically incident primary particles is related to the neutron intensity from primary particles incident over the entire upper hemisphere ([15]) according to

$$2\pi I_v = I(1 + x/L)\,,$$

where x is the atmospheric depth, L is the mean free path, I is the omnidirectional neutron intensity and I_v is the « vertical » neutron intensity.

Following the discussion given by SIMPSON ([15]), variations in intensity are related by

$$\frac{\delta I_v}{I} = \frac{\delta I}{I} - \frac{x/L}{1 + x/L} \cdot \frac{\delta L}{L}\,.$$

For neutron monitors at atmospheric depths $650\ \text{g/cm}^2 \leqslant x \leqslant 850\ \text{g/cm}^2$, where $L \approx 145\ \text{g/cm}^2$, this becomes

$$\frac{\delta I_v}{I_v} \approx \frac{\delta I}{I} - (0.84)\,\frac{\delta L}{L}\,.$$

SIMPSON and FAGOT ([16]) have shown that L is nearly a constant for atmospheric depths $x > 600\ \text{g/cm}^2$ and independent of latitude; thus we have

$$\frac{\delta I_v}{I_v} \approx \frac{\delta I}{I}\,.$$

([13]) J. A. SIMPSON, W. H. FONGER and S. B. TREIMAN: *Phys. Rev.*, **90**, 934 (1953).
([14]) W. H. FONGER: *Phys. Rev.*, **91**, 351 (1953).
([15]) J. A. SIMPSON: *Phys. Rev.*, **94**, 426 (1954).
([16]) J. A. SIMPSON and W. C. FAGOT: *Phys. Rev.*, **90**, 1068 (1953).

As a result, intensity variations observed with omni-directional neutron detectors located deep in the atmosphere are closely related to variations resulting from changes in the spectra of particles arriving near vertical incidence at the top of the atmosphere.

3·2. *Modulation.* – In calculating the latitude variation of the fractional change in neutron intensity due to modulation effects, the incident primary spectra were modified by the modulation factor given by PARKER (5); thus we have

$$j_z(R, N) = j_z(\infty, N) \frac{N^2}{N^2 + W_0 C(\infty)/Z^2},$$

where $j_z(R, N)$ is the primary rigidity spectrum near the earth, $j_z(\infty, N)$ is the galactic rigidity spectrum, W_0 is the rest energy of nuclei with charge Z, N is the magnetic rigidity pc/Z and $C(\infty)$ is a factor which depends only on the field strength and number density of magnetized gas clouds surrounding the earth. This modulation factor results from assuming that the scale length of scattering centers which make up the geocentric nebula is small compared to the Larmor radii of particles down to the lowest rigidities observed.

The neutron yield functions used in the calculations were chosen so that when combined with suitable primary spectra and the westward shifted geomagnetic co-ordinate system, a good fit was obtained between the calculated and observed latitude variation of neutron intensity at 680 g/cm² atmospheric depth (16). Thus, the yield function proposed by FONGER (14) was combined with the primary spectra due to KAPLON *et al.* (17) while a revised yield function (18) was combined with the primary spectra suggested by SINGER (19). Since both combinations give good fits with the latitude variation of neutron intensity, the modulation variations are essentially insensitive to the choice of yield function-primary spectra combination.

With the above combinations of primary spectra and yield functions, the latitude variation of $\delta I_v/I_v$ was calculated using a combination of graphical and analytical methods for $C(\infty) = 2.65$, 5.00 and 7.50 respectively. The value $C(\infty) = 2.65$ is that suggested by PARKER (5) so as to produce a cut-off below 1 GeV; the values $C(\infty) = 5.00$ and 7.50 were introduced to indicate the effects of denser nebulae of disordered fields.

The variations of $\delta I_v/I_v$ obtained in this manner, together with the amplitudes of the neutron intensity decreases observed at mountain altitudes between

(17) M. F. KAPLON, B. PETERS, H. L. REYNOLDS and D. M. RITSON: *Phys. Rev.*, **85**, 295 (1952).

(18) R. R. BROWN: *Nuovo Cimento* (in press).

(19) S. F. SINGER: *Progress in Elementary Particle and Cosmic Ray Physics* (New York, 1957), Vol. 4.

the geomagnetic equator and the knee of the latitude curve during the month of February, 1956, are shown in Fig. 1. The absicissa of this figure is expressed in terms of the vertical cut-off momentum for protons instead of geomagnetic latitude because of the longitude effect. In comparing $\delta I_v/I_v$ with experimental observations, it is assumed that the portions of the primary spectra giving rise to disintegration neutrons deep in the atmosphere were not too different from the quiescent spectra prior to the outbreak of strong solar activity and that modulation effects produced by the intense outburst of solar activity were considerably greater than the effects building up since the solar minimum in mid-1954.

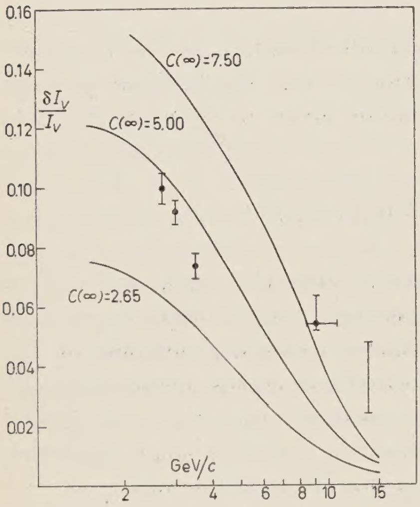

Fig. 1. – Fractional change in neutron intensity resulting from modulation effects as a function of cut-off momentum for vertically incident protons for monitors located deep in the atmosphere.

Inspection of Fig. 1 shows that the modulation factor obtained using $C(\infty) = 5.00$ yields variations in the vicinity of the knee of the latitude curve of the magnitude observed during the February, 1956 decrease. However, it is seen that the variations at equatorial latitudes for this value of $C(\infty)$ are too small, by at least a factor of two, to agree with experimental observations. In view of this result, it would seem necessary to examine Parker's mechanism for additional features which might reconcile the model with observations.

3·3. *Structure effects*. – A first step in this direction would involve considering the possibility that the geocentric nebula contained some scattering centers for which the scale length was comparable to, instead of smaller than, the Larmor radii of particles in the low rigidity range. As indicated by PARKER [5], the effect of including such reflecting-type centers in the nebula would be to depress the density of high rigidity particles more, for a given reduction of low rigidity particles, thus making the latitude variation of the fractional change in intensity less steep in going from high to low geomagnetic latitudes. Following this suggestion, the inclusion of a small contribution from reflecting centers results in the primary spectra taking the form

$$j_z(R, N) = j_z(\infty, N)\,\frac{N^2}{N^2 + \dfrac{W_0^2 C(\infty)}{Z^2}\left(1 - \dfrac{W_0^2 D(\infty)}{Z^2 C(\infty) N^2}\right)},$$

where the term $W_0^2 D(\infty)/Z^2 C(\infty) N^2$ in the denominator gives the effect of the reflecting centers. The constant $D(\infty)$, unlike $C(\infty)$, which depends only on the field strength and number density of scattering centers, involves the scale length $l(r)$ of the centers; thus, in Parker's notation,

$$D(\infty) = \frac{1}{R^2}\left(\frac{4R^2Z^2\Phi^2}{\pi^3 W_0^2}\right)^2 \int_R^\infty \frac{N(r)\,\mathrm{d}r}{r^2\pi l^2(r)}\,.$$

In view of this, it is difficult to determine $D(\infty)$ without making arbitrary assumptions as to the structure of the geocentric nebula. However, for a mixture of transmitting and reflecting centers where the effects of the transmitting centers predominate, we have $W_0^2 D(\infty)/Z^2 C(\infty) N^2 < 1$ for the lowest rigidities observed. Thus, for monitors at mountain altitudes where the yield function $S(E, x)$ vanishes near kinetic energies of the order of 1 GeV/nucleon, we find that $C(\infty)/D(\infty) > 0.4$. Calculations based on a nebula consistent with this approximation show that the effects of a small fraction of reflecting centers are confined largely to energies below 5 GeV/nucleon and are too small to be considered significant in the present instance.

The addition of a larger proportion of reflecting centers would reduce the high latitude variations beyond that considered above and extend the effects to energies above 5 GeV/nucleon. However, due to the uncertainty in the structure terms involving the scale length $l(r)$, an analytical form for the modulation factor is not readily available. Ultimately, when reflecting centers predominate, the modulation effect takes the form of a uniform depression of intensity.

It is important to note that there is a limiting feature in connection with reflecting centers which must be taken into account in attempting to use the effects of such centers to explain the variations observed in the particular period of solar activity under discussion. This feature is related to the unique superposition of cosmic ray effects (flare increase and 27 day decrease) observed in February, 1956. Thus, since the flare increase occurred approximately three days after the intensity minimum, the additional random scattering of particles due to reflecting centers being included in the nebula to account for the latitude variation of the decrease cannot be so great as to destroy the impact zone effects observed for flare particles coming from the sun a few days later.

3·4. *Geomagnetic effects.* – A second step in attempting to bring the calculations into agreement with observation would involve considering the possibility that the presence of a geocentric nebula of disordered magnetic fields produces geomagnetic perturbations, which in turn result in cosmic ray

intensity decreases. Justification for this view-point comes from two directions. First, from the experimental side, it has become evident from investigations of the cosmic ray geomagnetic co-ordinate system [11] that the earth's field at relatively large distances is seriously distorted, probably as a result of the interaction of magnetized gas clouds in the vicinity of the earth with the lines of force of the geomagnetic field. Second, from the theoretical side, it has been suggested that geomagnetic perturbations [20] result from the capture of interplanetary hydrogen and that cosmic ray variations [5] may accompany these field perturbations if the interplanetary hydrogen contains a mixture of magnetized and non-magnetized gas.

While the shift in cosmic ray co-ordinates is not understood at present the terrestrial capture of magnetized interplanetary hydrogen is thought to produce a compression of the lines of force of the geomagnetic field as a result of the weight of the gas bearing down on the field. The presence of non-magnetized gas, on the other hand, has just the opposite effect [20], inflating the lines of force above the top of the atmosphere by pushing upwards on the ionized gas in which the lines of force are embedded.

Since these two lines of argument suggest that geomagnetic perturbations might result from the interaction of magnetized gas clouds with the earth's magnetic field, calculations have been carried out to estimate the magnitude of the perturbations required to reduce the ratio of variations between latitudes near the knee and the equator. In doing this, it was assumed that the compression of the lines of force would result in uniformly increasing the cut-off rigidities for primary particles. By way of illustration of the effects which result, two calculations were made, one assuming that the geomagnetic perturbation accompanying a nebula for which $C(\infty)=5.00$ produces a 5% increase in cut-off rigidity and the other with $C(\infty)=2.65$ using a 2.5% increase. The results of these calculations are shown in Fig. 2, where the experimental data are also included. Inspection of this figure shows that increasing the cut-off rigidity uniformly has the greatest effect on the latitude variation

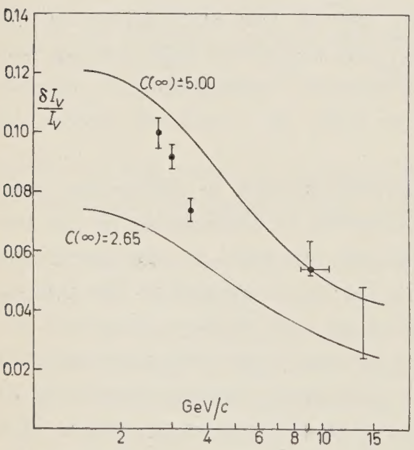

Fig. 2. – Fractional change in neutron intensity resulting from modulation effects and geomagnetic perturbations as a function of cut-off momentum for vertically incident protons for monitors located deep in the atmosphere.

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[20] E. N. Parker: *Journ. Geophys. Res.*, 62, 625 (1957).

of $\delta I_v/I_v$ at equatorial latitudes, thereby decreasing the ratio of knee to equatorial variations.

It should be noted that while geomagnetic perturbations of the type considered above produce cosmic ray effects similar to those resulting from the inclusion of reflecting centers in the geocentric nebula, the accompanying deflection of cosmic ray particles is not entirely of a random nature; rather, the geomagnetic field probably remains fairly orderly under compression and systematic deflections of the particles result. Thus, a shift, rather than a broadening, of the impact zones for flare particles would be expected.

4. – Discussion.

The experimental data used for comparison with the calculated latitude variation of intensity decreases was restricted to observations obtained from monitors at mountain altitudes ($650\ \text{g/cm}^2 \leqslant x \leqslant 850\ \text{g/cm}^2$) because at the present time the only available yield functions ([14,18]) were obtained from latitude curves at $680\ \text{g/cm}^2$ atmospheric depth ([16]). Recent measurements of the neutron latitude variation ([21]) at sea level indicate that the latitude variation between the geomagnetic equator and the knee of the latitude curve is a factor of 1.77, somewhat smaller than the factor of 2.55 at $680\ \text{g/cm}^2$ atmospheric depth. This indicates that the yield function appropriate to sea level calculations falls off more rapidly at low primary energies than the yield functions at mountain altitudes. As a result, intensity variations at low or intermediate altitudes (Albuquerque, New Mexico, and Mt. Washington, New Hampshire, for example) are smaller than would be observed at depths of the order of $700\ \text{g/cm}^2$ at the same geographical locations. Thus, the intensity decreases at these locations given in Table I represent lower limits for the variations at depths where the yield functions are fairly well known.

The principal deviation of the calculated curves in Fig. 1 is at low and equatorial geomagnetic latitudes. Thus, while the Huancayo ion chamber variation (2.4%) represents an *extreme* lower limit for neutron variations at this latitude, it is still at least a factor of two greater than that given by the calculations based on Parker's modulation mechanism. This discrepancy becomes more pronounced if the ion chamber variation is increased by a factor of two or so to give a more realistic estimate of the neutron variations. In addition, the neutron variation at Mr. Norikura, while also somewhat underestimated, is about a factor of two larger than that expected from the latitude

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([21]) D. C. Rose, K. B. Fenton, J. Katzman and J. A. Simpson: *Can. Journ. of Phsy.*, **34**, 968 (1956).

curve for $C(\infty) = 5.00$, which is reasonably close to the high latitude variations.

A latitude variation similar to that given above for the February, 1956, decrease was obtained by KODAMA and MIYAZAKI [22] in studying neutron monitor data for an isolated Forbush decrease which occurred on January 21, 1957. Since, with the geocentric nebula model, the Forbush decrease and the 27 day decrease differ only in the rate at which the magnetic barrier is built up, these observations also suggest that the nebula model does not give a good description of the latitude variation of cosmic ray intensity decreases.

As indicated in Sect. 3.2, structure effects, such as due to reflecting centers, set in first at high latitudes. While it is not possible to estimate the effects of a large proportion of reflecting centers without making arbitrary assumptions as to the structure of the nebula, it is clear that a rather dense nebula containing mostly reflecting centers would be required to yield low latitude variations of the order of 5% and at the same time, high latitude variations of the order of 10%. However, impact zone effects observed during the February 23 flare suggest that the random scattering due to the presence of a geocentric nebula was not much in excess of that estimated for a nebula containing only transmitting centers.

Geomagnetic perturbations, such as discussed in Sect. 3.3, have their principal effect at low latitudes. Thus, increases in proton cut-off momenta of the order of $2\frac{1}{2}$ to 5%, as seen in Fig. 2, increase the magnitude of variation at equatorial latitudes to the point where they become comparable with those observed while only slightly increasing the magnitude of variations at higher latitudes.

If such increases in cut-off momenta are to be attributed to compressional effects in the geomagnetic field, the question of the propagation of these effects down to the surface of the earth is raised at once, especially, in view of the magnitude of the geomagnetic perturbations required to explain the cosmic ray variations. In this connection, PARKER [20] has considered the geomagnetic effects due to the capture and diffusion of non-magnetic gas clouds into the earth's field; these calculations indicate that the horizontal component of the earth's field would decrease by as much as 0.2 per cent due to this mechanism. The geomagnetic effects due to a mixture of magnetic and non-magnetic clouds impinging on the earth's field have not been considered yet. However, from the discussion given by PARKER [5,20], both compressional effects, due to magnetic clouds, and inflation effects, due to non-magnetic clouds, would be expected. If the latitude variation of cosmic ray decreases are to be understood on this basis, it suggests that compressional effects are limited to regions at

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[22] M. KODAMA and Y. MIYAZAKI: private communication.

great distances from the earth and their propagation downward is hindered by non-magnetic gas inflating the geomagnetic field near the earth's surface.

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Note added in proof.

Prof. J. A. SIMPSON (private communication) has kindly provided additional data on neutron monitor intensity variations during February 1956. Thus, at Sacramento Peak, N.M., Mexico City, Mexico and Huancayo, Peru, neutron monitors recorded intensity decreases of 8.2%, 4.9% and 2.9%, respectively, during this period. In addition, the Climax monitor data indicate that just prior to the outbreak of intense solar activity in January 1956 the cosmic ray intensity was within 0.5% of the intensity at solar minimum, June 1954.

RIASSUNTO (*)

È stata calcolata la variazione delle diminuzioni d'intensità dei raggi cosmici in funzione della latitudine per l'effetto di modulazione prodotto da una nebulosa geocentrica di campi magnetici disordinati. Dati sperimentali raccolti per mezzo di osservazioni durante un periodo d'intensa attività solare dimostrano, se confrontati coi nostri calcoli, che le variazioni equatoriali superano per un fattore 2 o maggiore quelle date dall'agente modulante.

(*) *Traduzione a cura della Redazione.*

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Nucleon Structure and Pion-Phenomena.

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Summary. — The effect of the nucleon structure is discussed correlating phenomenologically a variety of pion-nucleon phenomena with the nucleon core. In the high energy region, the various experimental evidence is interpreted in terms of nucleon structure. The main results are: 1) The optical radius R of the proton estimated from the π^--p elastic scattering is nearly constant in the energy region $(1 \div 5)$ GeV. 2) The effective collision parameter in the π^--p reaction shows the nearly constant value $0.8 \cdot 10^{13}$ cm in the $(1 \div 5)$ GeV region. 3) The mean energy of the mesons produced in the π^+-p reaction in the energy range $(1 \div 5)$ GeV indicates a picture of meson production as a shaking off of mesons from the nucleon core. 4) In the nucleon-nucleon collision in GeV order, the mean neergy of the produced mesons suggests a shaking off of the external meson cloud such as given by the isobar model. 5) A regularity of the reduced multiplicity is discussed concerning the proton-antiproton annihilation star. In the low energy region, the well known failure of the meson theory on the S wave is to be ascribed to the internal effect of the nucleon structure (Appendix).

1. – Introduction.

As is well known, the low energy meson theory has showed a great success in the nuclear force (1) and P wave phenomena (2-4) such as π-nucleon scat-

(1) See for example, J. Iwadare, S. Otsuki, M. Sano, S. Takagi and W. Watari: *Prog. Theor. Phys.*, **16**, 455, 472, 604, 658 (1956).

(2) G. Takeda: *Phys. Rev.*, **95**, 1078 (1955); D. Ito, Y. Miyamoto and Y. Watanabe: *Prog. Theor. Phys.*, **13**, 594 (1955); M. H. Friedman, T. D. Lee and R. Christian: *Phys. Rev.*, **100**, 1494 (1956).

(3) G. F. Chew: *Phys. Rev.*, **89**, 591 (1953); S. Chiba, M. Yamazaki and N. Fukuda: *Prog. Theor. Phys.*, **12**, 767 (1954).

(4) G. C. Wick: *Rev. Mod. Phys.*, **27**, 239 (1955); G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1570, 1579 (1956).

tering or γ-π production in the $I=\frac{3}{2}$, $J=\frac{3}{2}$ state, but for the S wave scattering and anomalous magnetic moment a powerful and definite theory is still wanting. So far as P state is concerned, if an appropriate consideration is taken of the largeness of meson nucleon coupling, any method such as Intermediate coupling (2), Tamm-Dancoff (3) or the more precise Chew-Low (4) method gives qualitatively equal results in good agreement with experiment. This fact would fairly confirm the safe validity of the static extended source model in that problem. Nevertheless, with regard to the S wave scattering and anomalous magnetic moment (5) any above method could not yield good results, in spite of the same model and the same approximation. This situation should be put down only to the inadequacy of the approximation contained? Or, should it be ascribed to some unknown physical ground? We are now tempted conciously to adopt the latter point of view which should seem to shed some light on the cause of failure of the theory. We think that our ignorance or neglect of the internal structure of the nucleon have just caused the above difficulty of the theory. (See Appendix, where, on the basis of simplicity, taking an example of the static extended source theory the effects from the internal part of the cut-off region will be discussed. It would serve as a measure of the problem.) Then, in the problem such as S wave scattering where the unknown dynamics of nucleon structure seems to take its part essentially the point interaction theory, even relativistic, might be not so reasonable, even if it could obtain a nearly satisfactory agreement with experiment by the cut-off procedure. From such point of view the problems are separated in two parts where the nucleon structure is irrelevant or relevant. For the former, the meson theory may be considered, to that approximation, rather complete as so is quantum electrodynamics. For the latter, the unknown structure of the nucleon should be revealed step by step as the nuclear structure has been made clear patiently.

In the low energy phenomena the nucleon structure appears, say, rather in a implicit way, where the failure itself of the theory may reflect the unknown structure secretly. On the other hand, in relation to the high energy phenomena, as will be seen later, the experimental evidence on the nucleon core is rather explicit. The recent observation of the antiproton (6) will clearly indicate the fact that the proton is an elementary particle obeying the Dirac equation, while the electron-proton scattering (7) has manifestly revealed the existence of the

(5) See for example G. Salzman: *Phys. Rev.*, **105**, 1076 (1957).

(6) See for example, W. H. Barkas, R. W. Birge, W. W. Chupp, A. C. Ekspong, G. Goldhaber, S. Goldhaber, H. H. Heckman, D. H. Perkins, J. Sandweiss, E. Segrè, F. M. Smith, D. H. Stork, L. van Rossum, E. Amaldi, G. Baroni, C. Castagnoli, C. Franzinetti and A. Manfredini: *Phys. Rev.*, **105**, 1037 (1957).

(7) R. Hofstadter: *Rev. Mod. Phys.*, **28**, 214 (1956).

proton core. Possibly, the nucleon is the first elementary particle whose size has come into evidence. Then, the difficulty of the meson theory should be none other than that of the nucleon core, and without detailed knowledge of the latter the former would hardly be solved.

In view of the present stage where we have yet no systematic way of treating the structured nucleon, its phenomenological determination would be very suggestive as a first step to the question. Or rather, the first importance would be, besides others, to make out the unknown nucleon structure as far as possible, as while collecting all information on the interested subject. Namely instead of making assumptions about the specific form or character of the structure, it would be very suggestive to correlate a variety of pion-nucleon data in terms of the structured nucleon. Such approach consists of nearly considering the structure of the nucleon much as we might consider the structure of a nucleus through the data of the nuclear reaction, scattering, decay and so on (*). Along such lines, we shall undertake a phenomenological research on the evidence of the nucleon structure appeared in the recent high energy scattering experiments up to 5 GeV. In Sect. **2** and **3** the nucleon structure in π^--p scattering will be discussed, and Sect. **4** and **5** concern the nucleon-nucleon scattering.

By the way, after having somehow recognized the existence of the nucleon core, there may arise a doubt whether the nucleon core be of causal character or not. From the field theoretical point of view it would be an interesting question. This problem of causality is treated in a separate paper ([11]).

2. – π^--p collision at 5 GeV. ([12]).

The structure of the nucleon, especially the isobar cloud of (I, J, $=\frac{3}{2}$), played an important role in the nucleon-nucleon collisions in GeV region. In the π-nucleon collision in GeV order, the effect from the more internal structure has been expected, but the previous analysis ([3]) of the experiments at $1.0 \div 1.5$ GeV

(*) Some similar attitude, more or less phenomenological, has been taken towards the various pion problems by the several authors such as SACHS ([8]), TAKETANI ([9]), UMEZAWA ([10]) or others.

([8]) R. G. SACHS: *Phys. Rev.*, **87**, 1100 (1952); **95**, 1065 (1954).

([9]) M. TAKETANI, S. MACHIDA and S. ONUMA: *Prog. Theor. Phys.*, **6**, 638 (1951); **7**, 45 (1952).

([10]) H. UMEZAWA, Y. TAKAHASHI and S. KAMEFUCHI: *Phys. Rev.*, **85**, 505 (1952).

([11]) D. ITO, S. MINAMI and H. TANAKA: *Nuovo Cimento*, in press.

([12]) W. MAENCHEN, W. B. FOWLER, W. M. POWELL and R. W. RIGHT: *Phys. Rev.*, **108**, 850 (1957).

could not give a definite answer. Now given the data at 5 GeV we shall briefly analyse the quantities connected with the nucleon structure.

A-1. *Summary of the experimental result.* – The experimental data of π^--p collision at 5 GeV used for our following analysis are summarized.

1) Cross-section.

		Hydrogen filled Diffusion Cloud Chamber	Emulsion
(2.1)	$\sigma_{tot} =$	(22.5 ± 2.4) mb	28.7 mb
	$\sigma_{el} =$	(4.5 ± 1.0) mb	6.0 mb

2) Angular distribution of elastic scattering. For the hydrogen-filled diffusion cloud chamber data, the angular distribution in center of mass system is reproduced in the Fig. 1.

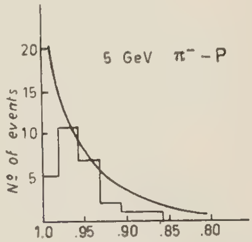

Fig. 1. – Angular distribution of π^--p elastic scattering at 5 GeV in a hydrogen-filled diffusion cloud chamber. The curve is given by $\mathrm{d}\sigma_{el}/\mathrm{d}\Omega = 1/(1.047 - \cos\theta)^2$ on the assumption of shadow scattering.

3) Recoil proton in inelastic scattering. In the center of mass system of the inelastic scattering the recoil proton hardly changes its direction. The mean momeutnm and energy are

$$\langle P \rangle \approx 0.9 \text{ GeV}, \qquad \langle E_p \rangle = 1.3 \text{ GeV}. \tag{2.2}$$

4) Angular correlation. It does not seem to exist any angular correlation between the final state particles. The Q-value is distributed rather isotropically without a tendency of focusing upon a certain value. The mean values of the momentum and energy of the produced mesons are:

$$\langle k \rangle \approx 0.60 \text{ GeV}, \qquad \langle \varepsilon \rangle \approx 0.62 \text{ GeV}. \tag{2.3}$$

A-2. *Conjecture of the optical parameter.* – If the elastic scattering can be considered as only a shadow scattering, it was shown that the following empirical relation holds ([13]):

$$\frac{\mathrm{d}\sigma_{el}}{\mathrm{d}\Omega} = \left(\frac{b/2}{a - \cos\theta}\right)^2 \tag{2.4}$$

([13]) D. Ito, T. Kobayashi, Y. Yamazaki and S. Minami: *Prog. Theor. Phys.*, **18**, 264 (1957).

where a is proved to correspond to the optical radius by the following relation

$$a = 1 + \left(\frac{2}{kR}\right)^2 .$$

Then, using the linear dependence of $(d\sigma_{el}/d\Omega)^{-\frac{1}{2}}$ on $(1 - \cos\theta)$ one may expect to determine the optical radius [14]. But the error of the angular distribution being large we can know only the limiting values

$$(2.5)\qquad 2.8\cdot 10^{-13}\ \text{cm} > R > 0.78\cdot 10^{-13}\ \text{cm} .$$

Next, we shall estimate R from the σ_{tot} and σ_{el}. As was shown in the previous works, if the angular distribution of elastic scattering is expressed by the eq. (2.4), the optical radius R is given by

$$(2.6)\qquad \pi R^2 = \frac{\sigma_{tot}^2}{2\sigma_{el}} - 2\pi\lambda\!\!\!^{-2} ; \qquad \lambda\!\!\!^{-} = \frac{1}{k} .$$

Using the hydrogen cloud chamber data in (2.1), R is estimated as

$$(2.7)\qquad R = 1.28\cdot 10^{-13}\ \text{cm} .$$

The error of this estimation possibly comes from the error of σ_{el}, because the latter is very difficult to determine exactly. Then, the above value $R = 1.28\cdot 10^{-13}$ cm would be in good agreement with $R = 1.16\cdot 10^{-13}$ cm obtained at $(1.0 \div 1.5)$ GeV. Accordingly, it may be safely said that the optical radius is nearly constant in the energy region $(1.0 \div 5.0)$ GeV.

With the above value of R, $R = 1.28\cdot 10^{-13}$ cm, the angular distribution (2.4) becomes

$$(2.8)\qquad \frac{d\sigma_{el}}{d\Omega} \approx \frac{1}{(1.047 - \cos\theta)^2} ,$$

which yields the distribution curve shown in Fig. 1.

Further, the opacity is evaluated by

$$(2.9)\qquad \alpha = \frac{\sigma_{inel}}{\pi R^2} = 0.34 .$$

A-3. *Effective angular momentum.* – If the shadow scattering amplitude is expressed by the eq. (2.4), using the partial wave analysis as in the previous work [14,15], one can calculate which incident waves are important in the

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[14] D. Ito, S. Minami and H. Tanaka: *Nuovo Cimento*, **8**, 135 (1958).

[15] D. Ito and S. Minami: *Prog. Theor. Phys.*, **14**, 198 (1955).

reaction. However, in order to know the most probable angular momentum in the reaction, the following sum rule may be simpler and more general.

If the real part (dispersive part) of the forward scattering amplitude is, as is the case for GeV scattering, much smaller than the imaginary part, the following relations hold for the cross-sections:

$$\left\{\begin{aligned} &\sigma_{\text{tot}} = 2\pi\bar{\lambda}^2 \sum_l (2l+1)(1-\eta_l) = \sum \sigma_l^{\text{tot}}\,, \\ &\sigma_l^{\text{tot}} = 2\pi\bar{\lambda}^2(2l+1)(1-\eta_l)\,, \end{aligned}\right. \tag{2.10}$$

$$A(x) = \left(\frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\Omega}\right)^{\frac{1}{2}} = \frac{\bar{\lambda}}{2}\sum_l (2l+1)(1-\eta_l)P_l(x)\,, \qquad x = \cos\theta, \tag{2.11}$$

where η_l shows the effect by absorption. In the special case $\theta = 0$, is

$$A(1) = \frac{\bar{\lambda}}{2}\sum (2l+1)(1-\eta_l) = \frac{\sigma_{\text{tot}}}{4\pi\bar{\lambda}} \tag{2.12}$$

namely, what is called the optical theorem:

$$\left(\frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\Omega}\right)_{\theta=0^\circ} = \left(\frac{k\sigma_{\text{tot}}}{4\pi}\right)^2. \tag{2.13}$$

In the vicinity of $x = 1$, $P_l(x)$ may be developed as

$$P_l(x) = 1 - \frac{l(l+1)}{2\cdot 1^2}(1-x) + \frac{(l+2)(l+1)l(l-1)}{1^2\cdot 2^2\cdot 2^2}(1-x)^2 + \dots. \tag{2.14}$$

Introducing (2.14) into (2.11) and differentiating with respect to x,

$$A'(1) = \frac{\bar{\lambda}}{2}\sum_l (2l+1)(1-\eta_l)\frac{l(l+1)}{2}\,. \tag{2.15}$$

Therefore,

$$\frac{A'(1)}{A(1)} = \frac{\sum\limits_l \sigma_l^{\text{tot}} l(l+1)/2}{\sum\limits_l \sigma_l^{\text{tot}}}\,. \tag{2.16}$$

In other words, the logarithmic derivative $A'(x)/A(x)$ at $x=1$ $(\theta = 0)$ gives the averaged values of $l(l+1)/2$ over the partial wave contribution σ_l^{tot}.

$$\left\langle\frac{l(l+1)}{2}\right\rangle = \frac{A'(1)}{A(1)}\,. \tag{2.17}$$

This is just the sum rule for our purpose. In the above deduction nothing being supposed on the form of the function, this relation is more general than the previous one.

If one supposes the amplitude $A(x)=(b/2)/(a-x)$ as in (2.4), the mean value of $l(l+1)/2$ becomes, for the value $R=1.28\cdot10^{-13}$ cm,

$$\left\langle\frac{l(l+1)}{2}\right\rangle=\left(\frac{kR}{2}\right)^2\approx 20\,, \tag{2.18}$$

from which, one can estimate $\langle l\rangle$ to an approximation

$$\left\langle\frac{l(l+1)}{2}\right\rangle\approx\frac{\langle l\rangle(\langle l\rangle+1)}{2}\,,$$

yielding the effective angular momentum

$$\langle l\rangle=6\,. \tag{2.19}$$

The value $R=1.28\cdot10^{-13}$ cm will be probably overestimated, and if the value $R=1.16\cdot10^{-13}$ cm from the $(1.0\div1.5)$ GeV data is used, the effective angular momentum is

$$\langle l\rangle=5\,. \tag{2.20}$$

A-4. *Effective collision parameter* $\langle d\rangle$. – The effective collision parameter $\langle d\rangle$ is obtained from the relation $k\langle d\rangle=\langle l\rangle$. As was seen above, l being rather large one can approximate the eqs. (2.17) and (2.18) as

$$\sqrt{\left\langle\frac{l(l+1)}{2}\right\rangle}\approx\sqrt{\frac{(\langle l\rangle)^2}{2}}=\sqrt{\left(\frac{kR}{2}\right)^2}\,. \tag{2.21}$$

Then, one obtains the reasonable relations,

$$\left\{\begin{aligned}\langle l\rangle&\approx\frac{kR}{\sqrt2}\,,\\ \langle d\rangle&\approx\frac{R}{\sqrt2}\,.\end{aligned}\right. \tag{2.22}$$

From the above relations, $\langle d\rangle$ is evaluated for the various values of the optical parameter R,

$$\left\{\begin{array}{lll}0.9\ \cdot10^{-13}\text{ cm} & \text{for} & R=1.28\cdot10^{-13}\text{ cm}\,,\\ 0.82\cdot10^{-13}\text{ cm} & & R=1.16\cdot10^{-13}\text{ cm}\\ 0.71\cdot10^{-13}\text{ cm} & & R=1.00\cdot10^{-13}\text{ cm}\end{array}\right. \tag{2.23}$$

This estimation shows that the π^--p collision at 5 GeV takes place at a distance much shorter than the meson Compton wave lenght $\hbar/\mu c = 1.4\cdot 10^{-13}$ cm ([16]). Or rather, one might conclude that the π^--p collision at 5 GeV occurs, as a rule, very close to the proton radius ([7]) $\langle r_p \rangle$,

$$\langle r_p \rangle = 0.7\cdot 10^{-13}\ \text{cm}\,. \tag{2.4}$$

3. – Nucleon structure and π^--p collision in the GeV energy region.

In the previous section, it was shown that the π^--p collision at 5 GeV happens near the proton radius. Such situation is nearly the same even in the case of 1.0 and 1.5 GeV. In the Table I the results of our previous analysis for 0.7, 1.0, 1.5 and 4.5 GeV are given together.

TABLE I.

E^{Lab} (GeV.)	σ_{tot} (mb)	$\sigma_{\text{el}}/\sigma_{\text{inel}}$	R (10^{-13} cm)	$\langle d \rangle$ (10^{-13} cm)
0.7	43	1.6	0.89	0.63
1.0	46	1.0	1.14	0.81
1.5	30	0.5	1.18	0.83
4.5	20	0.25	1.16	0.82
5.0	22.5	0.26	1.28	0.90

Except the 0.7 GeV case for which the shadow scattering approximation is not so reasonable, the effective collision parameter will be, on the whole,

$$\langle d \rangle = 0.8\cdot 10^{-13}\ \text{cm}\,. \tag{2.25}$$

Thus, we would not commit a large error saying that *the* π^--p *collision in the GeV energy region is a collision with the proton core.*

There is also another evidence which supports the above conclusion strongly. It is the mean energy of the produced mesons.

([16]) It will be interesting to note that even in the low energy scattering Fermi remarked from the analysis of P phase that the interaction range may be $r_0 \sim 0.7(\hbar/\mu c)$ See E. FERMI: *Suppl. Nuovo Cimento*, **2**, 44 (1956).

The Fermi theory ([17]) failed to explain the mean momentum $\langle k \rangle$ of the produced mesons in the center of mass system. As an example, in the Fig. 2, the momentum distribution is given for the 1.5 GeV case. Fermi's theory predicting a k value much higher than the experimental one, the multiplicity $\langle n \rangle = E_0/\langle \varepsilon \rangle$ expected from the statistical theory is smaller than the observed value. In order to obtain agreement with the experiment his volume Ω must be taken rather large. Such situation is the same also for the 5 GeV case.

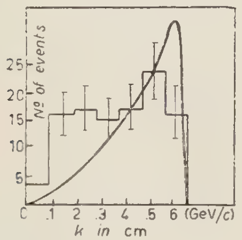

Fig. 2. – Momentum distribution in center of mass system of produced mesons by π^--p collision at 1.5 GeV. The curve is a prediction by the statistical theory.

The approximate values of mean momentum and energy $\langle k \rangle$ and $\langle \varepsilon \rangle$ in the center of mass system are given in Table II for $(1.0 \div 5.0)$ GeV collisions. (Such numerical values being not given in the experimental papers a rough estimation is done from the histograms).

TABLE II.

E^{Lab} (GeV)	$\langle k \rangle$ (GeV)	$\langle \varepsilon \rangle$ (GeV)	$\varepsilon_c \equiv \langle \varepsilon \rangle \sqrt{1-(v_0/c)^2}$ (GeV)
1.0	0.3	0.33	0.28
1.4	0,4	0.43	0.36
1.5	0.4	0.43	0.36
5.0	0.6	0.62	0.34

The last column gives the defined quantity $\varepsilon_c \equiv \langle \varepsilon \rangle \sqrt{1-(v_0/c)^2}$, $\langle \varepsilon \rangle$ multiplied by the Lorentz factor $\sqrt{1-(v_0/c)^2}$ of the center of mass system with respect to the laboratorium system. This quantity shows nearly the constant value

$$\varepsilon_c \approx 0.34 \text{ GeV} . \tag{2.6}$$

The reason to take account of such quantity is as follows. If the multiple production process could be visualized as a shaking off of wave packets from the proton eigenfield, the mean energy $\langle \varepsilon \rangle$ in the center of mass system

([17]) E. FERMI: *Prog. Theor. Phys.*, **5**, 570 (1950); *Phys. Rev.*, **81**, 683 (1951).

should be given by the Lorentz transformation of the mean packet energy ε_c in the proton rest system

$$\langle\varepsilon\rangle = \frac{\varepsilon_c}{\sqrt{1-(v_0/c)^2}}\,. \tag{2.27}$$

Then, if the picture of the « shaking off » is right, $\varepsilon_c = \langle\varepsilon\rangle\sqrt{1-(v_0/c)^2}$ will be expected to be constant independently of the incident pion energy, because ε_c should be the mean energy of the eigenfield of the proton at rest. From Table II, the mean energy and momentum of pions in the eigenfield of the proton at rest will be

$$\left\{\begin{aligned}&\varepsilon_c \approx 0.34\ \text{GeV}\,,\\ &k_c = \sqrt{\varepsilon_c-\mu^2} \simeq 0.31\ \text{GeV} \approx 2.2\,\mu\,.\end{aligned}\right. \tag{2.28}$$

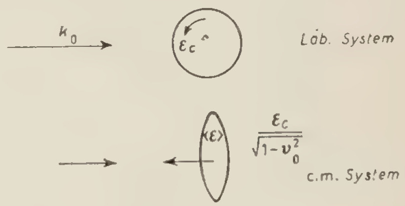

Fig. 3. – If ε_c is the mean energy in the eigenfield of the proton at rest, the mean energy in the center of mass system will be $\langle\varepsilon\rangle = \varepsilon_c/\sqrt{1-v_0^2}$, where v_0 is the velocity of the center of mass system to the laboratory system.

Hence, the order of extension r_c of the wave packet may be estimated by the uncertainty relation

$$r_c \approx \frac{\hbar}{k_c} \approx \frac{\hbar}{2.2\mu} \approx \langle r_p\rangle \tag{2.29}$$

in strong support of the picture that the *nucleon core has much to do with the multiple production process in the GeV energy region collision* [18].

4. – Nucleon-nucleon collision in the GeV energy region.

In our previous work [14] the proton « size » $r_p = 0.7\cdot10^{-13}$ cm was derived by the analysis of the nucleon-nucleon collision in the GeV energy region. Here, other relations will be treated on the data of nucleon-nucleon collision.

A-1. *Inelasticity K*. – In the center of mass system of the nucleon-nucleon collision, the inelasticity K is defined by

$$K = \frac{\text{Available energy}}{\text{Max. of available energy}} = \frac{E_0 - E_p}{E_0 - M}\,, \tag{3.1}$$

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[18] As regards this matter, according to our preliminary investigation, a reinterpretation of Kovacs' model in terms of « core » instead of the π-π interaction is very suggestive. There may be found a possibility of an unified interpretation of both the low and high energy phenomena. This point will be discussed in a forthcoming paper. J. S. Kovacs: *Phys. Rev.*, **93**, 252 (1954).

where E_0 is the total energy, E_p the recoil energy and M the nucleon mass. In the GeV events, E_0 and $\langle E_p \rangle$ are measured rather precisely ($\langle\ \rangle$ means the mean value). The numerical value is given in Table III where $\langle p \rangle$ was estimated from the histogram ([19]).

TABLE III.

T^{Lab} (GeV)	$\langle p \rangle$ (GeV)	$\langle E_p \rangle$ (GeV)	$\langle K \rangle$
0.81	0.35	0.95	0.86
0.93	0.38	1.00	0.70
1.50	0.65	1.13	0.43
2.75	0.90	1.29	0.33

A-2. *Reduced multiplicity.* – In the center of mass system, the conservation law holds for the production of n_k pions of energy ε_k,

$$2(E_0 - E_p) = 2K(E_0 - M) = \sum_k n_k \varepsilon_k = n\langle \varepsilon \rangle \,, \tag{3.2}$$

where $\langle \varepsilon \rangle$ is the mean energy and n the total number of mesons. Then, if one knows the mean energy $\langle \varepsilon \rangle$, the quantity n/K called « reduced multiplicity » is easily calculated,

$$\frac{n}{K} = \frac{2(E_0 - M)}{\langle \varepsilon \rangle} \,.$$

Recently in the cosmic ray data, KANEKO *et al.* ([20]) have found some regularity of the reduced multiplicity as a function of E_0. To understand the reason of such regularity we shall estimate the reduced multiplicity also for our GeV events. The estimation being rather difficult for $\langle n \rangle$, the mean energy $\langle \varepsilon \rangle$ is first evaluated and n/K is calculated by the eq. (3.3). For convenience of comparison with the data of KANEKO *et al.*, charge independence is supposed. Then,

$$\frac{\langle n_\pm \rangle}{K} = \frac{4(E_0 - M)}{3\langle \varepsilon \rangle} \,, \tag{3.4}$$

([19]) For example, W. B. FOWLER, R. P. SHUTT, A. M. THORNDIKE, W. L. WHITTEMORE, V. T. COCCONI, E. HART, M. M. BLOCK, E. M. HARTH, E. C. FOWLER, J. D. GARRISON and T. W. MORRIS: *Phys. Rev.*, **103**, 1489 (1956).

([20]) S. KANEKO and M. OKAZAKI: *Nuovo Cimento*, **8**, 521 (1958).

where $\langle n_{\pm} \rangle / K$ is the reduced multiplicity for charged pions. The values are given in Table IV.

TABLE IV.

T^{Lab} (GeV)	$\langle k \rangle$ (GeV)	$\langle \varepsilon \rangle$ (GeV)	$\langle n_{\pm} \rangle / K$	$\gamma_c - 1 = (E_0 - M)/M$
0.81	0.20	0.25	0.89	0.18
1,50	0.25	0.29	1.63	0.38
2.75	0.30	0.33	2.18	0.58

The Fig. 4 is a plotting of our data superposed on the figure given by KANEKO *et al.* The points derived from the GeV collision are on the line of tangent 0.78, a prolongation of which passes, as in the case for the cosmic ray data, through the proton-antiproton annihilation point. The meaning of such regularity is not clear for the cosmic ray case, but for the GeV collision case it will be rather comprehensible. That is, the regularity of the mean energy $\langle \varepsilon \rangle$.

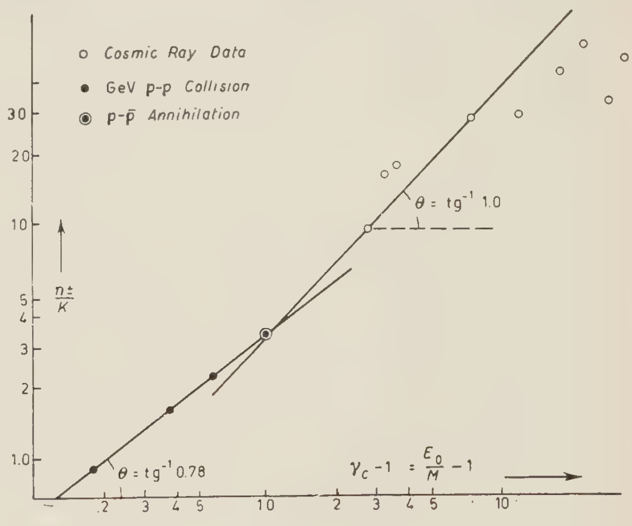

Fig. 4. – Reduced multiplicity plotted versus Lorentz factor $\gamma_c - 1$. Cosmic ray data were given by KANEKO *et al.*

5. – Nucleon structure and nucleon-nucleon collision in GeV.

B-1. *Multiple production of mesons.* – In the nucleon-nucleon collision case also, Fermi's theory predicts higher values for the mean energy. Fig. 5 is an example of such discrepancy. The reason is rather simple. In nucleon-nucleon collision of GeV order, the isobar state playing an important role, the phase volume contribution is less effective than the matrix element contribution.

In the same way as for the π^--p scattering, $\varepsilon_c = \langle\varepsilon\rangle\sqrt{1-(v_0/c)^2}$ is calculated and shown in Table V.

TABLE V.

T^{Lab} (GeV)	0.81	1.50	2.75
$\varepsilon_c \equiv \langle\varepsilon\rangle\sqrt{1-(v_0/c)^2}$ (GeV)	0.21	0.21	0.21

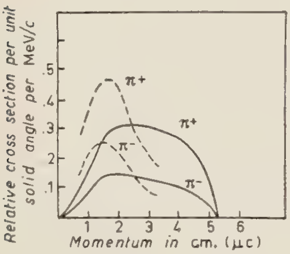

Fig. 5. – Relative cross-section of meson by p-p collision in GeV order. The dotted lines are experimental and real lines are predicted by the statistical theory.

That is,

$$(3.5)\qquad \begin{cases} \varepsilon_c = 0.21\ \text{GeV}\,, \\ k_c = \sqrt{\varepsilon_c^2 - \mu^2} \approx 1.1\ \mu\text{m}\,. \end{cases}$$

Hence, in GeV collision of nucleon-nucleon, the wave packet r_c of the order of

$$(3.6)\qquad \gamma_c \approx \frac{h}{k_c} \approx \frac{h}{\mu_0}\,,$$

mainly answers for the shaking off process of production. This indicates that the shaking off of the meson cloud is rather exterior as pictured in the isobar theory ([21]).

In nucleon-nucleon collisions of the GeV order, as was seen above, $\langle\varepsilon\rangle \approx \gamma_c\varepsilon_c$ having very weak dependence on the energy the reduced multiplicity changes almost linearly with the energy. However, such reasoning cannot be generalized to the regularity observed in the cosmic ray events higher than 10 GeV. Because in such high energy events the isobar effect would be much less probable. Further, if one supposed the linearity coming of $\varepsilon_c = \gamma_c\varepsilon_c$, the large multiplicity could not be explained yielding the small value for the extremely high energy events:

$$(3.7)\qquad \frac{\langle n_\pm\rangle}{K} = \frac{4M}{3\varepsilon_c}\left(1-\frac{1}{\gamma_c}\right) \to \frac{4M}{3\varepsilon_c} = 6 \qquad (\gamma_c \to \infty).$$

Therefore, the reason of the regularity in the cosmic ray events must be searched in another source.

However, in the cosmic ray events $\langle\varepsilon\rangle$ would seem to satisfy the following empirical relation (see Table VI),

$$(3.8)\qquad \varepsilon \approx 0.3\,\gamma_c^{\frac{1}{3}}\,.$$

([21]) S. J. LINDENBAUM and R. M. STERNHEIMER: *Phys. Rev.*, **105**, 1874 (1957).

TABLE VI.

T^{Lab}	5	10	10^2	10^3	10^4	10^5	10^6
γ_c	1.9	2.4	7.5	20	70	210	900
$\langle\varepsilon\rangle$ (GeV)	0.4	0.4	0.4	0.85	1.04	2.44	3.00
$\langle\varepsilon\rangle/\gamma_c^{\frac{1}{3}}$	0.32	0.30	0.22	0.49	0.25	0.41	0.31

According to the theory of Louis-Oppenheimer-Wouthuysen [22] (briefly L-O-W), the mean energy is expected to have the form

$$\langle\varepsilon\rangle \sim (\gamma_c K)^{\frac{1}{3}} . \tag{3.9}$$

When $K > 0.5$, $K^{\frac{1}{3}}$ being practically equal to 1, $\langle\varepsilon\rangle$ would be nearly $\langle\varepsilon\rangle \propto \gamma_c^{\frac{1}{3}}$. At that time the above tendency in eq. (3.8) may support L-O-W theory? Probably so, because in the higher energy region than 5 GeV the isobar effects are overwhelmed by the phase volume contribution and $\langle n_{\pm}\rangle$ depends essentially only on the invariant phase volume as expected in L-O-W theory. Then, the reduced multiplicity is given by

$$\frac{\langle n_{\pm}\rangle}{K} \propto K^{-\frac{1}{3}}\gamma_c^{\frac{2}{3}} .$$

If one takes account of the large error in the cosmic ray data, L-O-W's relation $\langle n_{\pm}\rangle/K_{\frac{2}{3}} \propto \gamma_c^{0.7}$ may not be so far from the observed results.

B-2. *Comments on proton-antiproton annihilation.* – It seems rather strange that the plots of the reduced multiplicity $\langle n_{\pm}\rangle/K$ both for the GeV collision and cosmic events pass through the point of proton-antiproton star. It might have some unknown reasoning, but we shall note the possibility of an accidental coincidence.

According to BARKAS *et al.* [6], the multiplicity by p-$\bar{\text{p}}$ annihilation is $\langle n\rangle \approx 5$. There, the mean energy $2M/5 = 0.39$ GeV shows a value accidentally close to that obtained from the nucleon-nucleon collision with $\gamma_c = 2$, *i.e.*, $\langle\varepsilon\rangle = \varepsilon_c\gamma_c = 0.2/\gamma_c$ GeV $= 0.42$ GeV. Therefore, the extrapolation of the multiplicity in the isobar model to the 6 GeV region might have coincided with that of the p-$\bar{\text{p}}$ star.

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(22) H. V. LEWIS, T. R. OPPENHEIMER and S. A. WOUTHUYSEN: *Phys. Rev.*, **73**, 127 (1943); T. D. LEE and E. M. HENLEY: *Phys. Rev.*, **101**, 1536 (1956).

The mean energy $\langle\varepsilon\rangle$ in p-$\bar{\text{p}}$ stars was found experimentally as

$$\langle\varepsilon\rangle = 0.322 \text{ GeV},$$

which is nearly the same as that of π^--p scattering in GeV, $\varepsilon_c = 0.34$ GeV. Probably because, the multiple production by p-$\bar{\text{p}}$ annihilation would be an affair of the proton core as discussed by KOBA and TAKEDA ([23]). On the other hand, the multiple production by nucleon-nucleon collision will be, as was seen above, a shaking off of the external meson cloud. Then, there must hardly be a relation between such different mechanisms.

However, if the reduced multiplicity $\langle n_\pm\rangle/K$ were determined only by the meson proper field character independently of the detailed creation mechanism, as is the case of the thermal radiation known as Kirchhoff's law, it should arise another interesting question. In the strong and complicated interactions such as predicted by the non linear or final interactions, there might realize a state which yields the linearity of the reduced multiplicity. Though such possibility cannot be excluded an accidental coincidence will be rather probable.

6. – Concluding remarks.

The effects of the nucleon structure were discussed correlating a variety of pion-nucleon data with the nucleon core. On the concept of the nucleon core one would understand the various aspects of pion phenomena rather comprehensively. In the low energy region the failure of the meson theory on the S wave will be ascribed to the internal effect of the nucleon « core » (Appendix). There, the failure itself of the theory reflecting the unknown mechanics of the internal structure would be an important clue for the future theory.

In the high energy region the several experimental evidences were interpreted in terms of nucleon core. The main results are:

1) The optical radius R of the proton estimated from the π^--p elastic scattering is nearly constant in the energy region $(1\div5)$ GeV.

2) The effective angular momenta $\langle l\rangle$ in π^--p reactions are: $\langle l\rangle = 2$ at 1.5 GeV and $\langle l\rangle = 5\div6$ at 5 GeV, but the corresponding collision parameter $\langle d\rangle$ shows nearly constant value in the $(1\div5)$ GeV region, $\langle d\rangle = 0.8\cdot10^{-13}$ cm.

3) The mean energy $\langle\varepsilon\rangle$ of produced mesons in π^--p reaction is smaller than expected by the statistical theory. In the center of mass system there

([23]) Z. KOBA and G. TAKEDA: *Prog. Theor. Phys.*, **19**, 269 (1958).

exist a relation for the mean energy. $\varepsilon_c = \langle\varepsilon\rangle\sqrt{1-(v_0/c)^2} \approx 0.34$ GeV (nearly constant independent of the energy), where v_0 is the velocity of the center of mass system. This regularity, together with the constancy of the effective collision parameter $\langle d\rangle = 0.8\cdot 10^{-13}$ cm will indicate a picture of meson production as the shaking off of mesons from the nucleon core.

4) In the nucleon-nucleon collision case also, one finds a relation for the mean energy of produced mesons in the center of mass system. $\varepsilon_c = \langle\varepsilon\rangle\cdot \cdot\sqrt{1-(v_0/c)^2} \approx 0.2$ GeV (constant nearly independent of the energy). This will suggest the shaking off of the external meson cloud as given by the isobar model.

5) The plotting of the logarithm of reduced multiplicity in nucleon-nucleon reaction versus the Lorentz factor $\gamma_c - 1 = (1/\sqrt{1-(v_0/c)^2}) - 1$ shows a line of tangent 0.78, whose prolongation passes through the point of the proton antiproton annihilation. The reason of the latter fact is not clear but is probably an accidental coincidence.

Though we scarcely know anything about the nucleon core structure a more realistic meson theory should be constructed, without doubt, on the basis of the structured nucleon. It is hoped then that our analysis give some insight into the reason for the qualitative success of the more specific theories and that our approach would make possible to obtain a fairly detailed quantitative interpretation in terms of nucleon core which influences other phenomena also. Our attitude is more phenomenological and less fundamental, but it will offer a procedure for giving a physical interpretation of the pion phenomena. In any way however, it should turn out, sooner or later, that the nucleon core is so complicated as to require discussion in terms of its detailed structure rather than in terms of a solution of a simple dynamical problem. If the nucleon core should really play an important role in both the low and high energy phenomena, an unified interpretation would become possible in terms of core ([18]) and the source of the difficulty of the present fundamental theory would be made clear by the detailed knowledge of the nucleon core.

Appendix

The effect from the internal part of the « core » of the « cut-off region » will be discussed in a quantitative way. Hereafter, we use the term « core » as a synonym of « cut-off region ». Therefore, such « core » may sometimes be different from that used in the text where the « core » means itself literally. For simplicity we shall take the static extended source theory as an example.

Then, in the configuration space, the pion-nucleon scattering is described in general by the following integro-differential equation:

$$(\mathrm{A.1}) \qquad (\nabla^2 + k_0^2)\varphi(\boldsymbol{x}, \zeta) = \int\limits_{|x'|>a} \mathrm{d}\boldsymbol{x}' \langle \boldsymbol{x}\zeta | V | \boldsymbol{x}'\zeta' \rangle \varphi(\boldsymbol{x}'\zeta') ,$$

where k_0 is the incident meson momentum, V the interaction kernel, $\boldsymbol{\zeta}$ a parameter representing the pion and nucleon states, and a is the so called cut-off radius. In terms of the fixed meson theory the variable $\boldsymbol{x}$ should be called the meson co-ordinate. There, the cut-off in the $\boldsymbol{x}$ space is proved to correspond to a special cut-off function in the momentum space,

$$(\mathrm{A.2}) \qquad \delta(\boldsymbol{K}) - \frac{a}{2\pi^2 k^2}\,[\sin ka - ka \cos ka] ,$$

which cuts off the region $k \gg 1/a$. Accordingly, if a is taken as $a \approx 1/k_{\max}$, the above equation is nearly equivalent to the ordinary cut-off theory ([24]) in the momentum space. The cut-off radius a would be taken physically as the order of the nucleon size inside of which nothing is known about the forces acting on the mesons entered there, because there must be dominating the *unknown* mechanics of the strange particles, nucleon pairs and so on. In view of this, the necessity of a « cut-off » of such complicated region has been earnestly insisted on. However, the philospohy of a « cut-off » must not be taken as an arrogant pretext for a « cut-down » of the *unknown* region but be a modest acknoledgement of our ignorance. What we would insist here on is the latter, our complete ignorance of the internal structure of the nucleon.

We « confess » our ignorance in the following way. All our knowledge about the inside of the cut-off region be represented by the boundary condition on the spherical surface $r = a$ of the « core ». That is, as was often done in the nuclear theory, the region $r < a$ is represented by the logarithmic derivative of the pion wave function at $r = a$ (*). This is why we choose the $\boldsymbol{x}$ space in place of the $\boldsymbol{K}$ space. The phase shift δ will be calculated for the given logarithmic derivative γ. If the phase shift depends critically on γ, the nucleon structure possibly plays an important role in that reaction. In the case that the phase shift is insensitive to the internal dynamics, the scattering may be called « shape independent ».

Separating the S-wave part aside, the radial wave function $f_l^{IJ}(r)$ for $l \geqslant 1$ satisfies the following equation:

$$(\mathrm{A.3}) \qquad \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k_0^2 - \frac{l(l+1)}{r^2}\right] f_l^{IJ}(r) = \lambda_l^{IJ} \int\limits_a^\infty V_l(r, r')\,\mathrm{d}r' f_l^{IJ}(r') ,$$

([24]) G. F. Chew: *Phys. Rev.*, **95**, 1669 (1954); N. Fukuda: Lecture Note in the Tokyo University of Education (1954).

(*) Such manner itself of « confession » clearly manifests our complete ignorance about the cut-off region or its meaning. Even for the non-relativistic case described by the eq. (A.1) the internal effects might be of relativistic character, where one is no more sure that such spacial separation of « inner » and « outer » parts shlould have a real meaning. As an extreme case, if the cut-off were due only to an acausal core the above approach should lose its meaning at all.

where λ_l^{IJ} are some statistical factors for the states of I, J, and I and J are the total spin and angular momentum respectively. In order to get the phase shift δ_l^{IJ} one must solve the above equation under the boundary conditions:

$$\text{(A.4)}\qquad \begin{cases} f_l^{IJ}(r) \xrightarrow[r\to\infty]{} \sin\left((k_0 r - \dfrac{\pi l}{2}\right) + \operatorname{tg}\delta_l^{IJ}\cos\left(k_0 r - \left(\dfrac{\pi l}{2}\right)\right), \\[2ex] a\,\dfrac{\mathrm{d}}{\mathrm{d}r}\ln f_l^{IJ}(r)\Big|_{r=a} = \gamma_l^{IJ}\,. \end{cases}$$

However, our purpose consists of obtaining the knowledge of the internal effects on the phase shift δ_l^{IJ}, and not of solving the equation mathematically. With this intention in mind, we shall approximate the equation in such a way that the essential features of the solution be seen easily.

As CHEW showed, the interaction kernel $V(r, r')$ is nearly separable, and by a suitable choice of $g_l(r)$ (*) the seaprated kernel

$$\text{(A.5)}\qquad V_l(r, r') = g_l(r)\,g_l(r')$$

could reproduce the pion nucleon interaction rather well. For our purpose however, such approximation is of no importance, and the equation might have been solved, if necessary, even by the Chew-Low method.

Under the approximation of the separated kernel, the eq. (A.3) allows a rigorous solution by the method of the variation of parameter. But for an incident energy lower than the P wave resonance (200 MeV) the meson momentum k_0 being $k_0 \lesssim 2\mu$ (in the center of mass system), our phase shift relation is approximated in the following form (the suffixes I and J are omitted):

$$\text{(A.6)}\qquad \begin{cases} \dfrac{\eta^{2l+1}}{(2l+1)!!\,(2l-1)!!}\operatorname{ctg}\delta_l = \\[2ex] \qquad = \dfrac{\dfrac{\eta^{2l+1}\operatorname{ctg}\delta_l^c}{(2l+1)!!\,(2l-1)!!} + \left(\dfrac{a}{\mu r_l^2}\right)^{2l+1}\xi_l}{1 - \left[\dfrac{(\mu a)^{2l+1}}{(2l+1)!!\,(2l-1)!!}\eta^{2l+1}\operatorname{ctg}\delta_l^c + 2\left(\dfrac{a}{r_l}\right)^{2l+1}\right]\xi_l}, \\[2ex] \eta = \dfrac{k_0}{\mu}, \\[2ex] \xi_l \approx \dfrac{\gamma_l - (l+1)}{\gamma_l + l}, \end{cases}$$

where $\operatorname{ctg}\delta_l^c$ is just what was given by Chew's theory, and r_l represents the spreading order of the interaction kernel $g_l(r)$. Therefore all the terms other than $\operatorname{ctg}\delta_l^c$ is the correction coming from the internal structure in a certain sense. Such correction terms depend, as expected naturally in our approach, on the magnitude of ξ_l and also on the relative order of the cut-off radius a to the extension of the kernel a/r_l. If the spread of the kernel $g_l(r)$ is of the

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(*) For the case of $l=1$, Chew's $g_e(r)$ is $\sim K_0''(\mu r)$, where K_0 is the modified Bessel function.

order of the Compton wave length, $\langle r_l \rangle \approx 1/\mu$, the phase shift relation is

$$\text{(A.7)} \qquad \frac{\eta^{2l+1}\operatorname{ctg}\delta_l}{(2l+1)!!\,(2l-1)!!} = \frac{\dfrac{\eta^{2l+1}\operatorname{ctg}\delta_l^c}{(2l+1)!!\,(2l-1)!!} + (\mu a)^{2l+1}\xi_l}{1-(\mu a)^{2l+1}\left[\dfrac{\eta^{2l+1}\operatorname{ctg}\delta_l^c}{(2l+1)!!\,(2l-1)!!}+2\right]\xi_l}$$

In the low energy events the effective range for the P wave being of order 10, the value of

$$\frac{\eta^{2l+1}\operatorname{ctg}\delta_l^c}{(2l+1)!!\,(2l-1)!!}\,,$$

must be of the same order. Then, in the nominator of the right hand side of the eq. (A.7) the coefficient $(\mu a)^{2l+1}$ of the correction term ξ_l becomes relatively very small even if the cut-off is done at the nucleon core $a \approx 1/2\mu$, that is, $(\mu a)^{2l+1} \approx (1/2)^{2l+1}$. If $a \approx 1/M$ as in the normal case of the cut-off theory, the coefficient will be much smaller $(\mu a)^{2l+1} \approx (1/7)^{2l+1}$. Consequently, the larger the angular momentum the smaller the correction coming from the internal structure (*).

Of course, ξ_l can take a value from $-\infty$ to $+\infty$, but as in the case of the nuclear force if the wave function in the « inside » region $r < a$ can be approximated by the repulsive potential, the logarithmic derivative γ_l is always positive, hence, $|\xi_l| = |(\gamma_l - (l+1))/(\gamma_l + l)| \sim 1$. Therefore, if the inside structure is repulsive against the pion wave, the higher the l value the smaller the internal effect because of the large barrier effect of the repulsive force. On the contrary, in the case of an attractive potential, the pion waves are pulled in and folded in the « core », and γ_l becomes negative. If $\gamma_l = -l$, this is just a case of « core resonance », and the cut-off theory loses its meaning at all. According to the recent electron-proton scattering (7) the meson current distribution is not so singular as the one of Yukawa, but rather repulsive preventing the dense distribution inside the core. If so, the incident meson of high l value will hardly penetrate into the inside of the nucleon, in strong support of our point of view that because of the non-overlapping of the P-wave with the « core » or the « internal region » prevented by the barrier effect of the centrifugal force the cut-off theory was a good approximation for the P-wave phenomena.

However, for the S-wave which penetrates easily into the « core » the effect of the internal structure is expected to be rather large, and this would be why all the theories without consideration of internal structure have failed to explain the S-wave scattering. On the basis of simplicity we discussed here, as an example, only on the static cut-off theory. But even with the covariant form taking into account the field reaction or nucleon pairs, though better revised, neither the point particle (25) nor the cut-off (26) could succeed in

(*) One may visualize this situation intuitively. In terms of classical picture, the overlapping of the wave (angular momentum l, wave length $\lambda\!\!\!^{-}$) with a source of radius a will be characterized by $(a/\lambda\!\!\!^{-})^{2l+1}$. Then, the larger the l-value the smaller the overlapping which represents a measure of the internal effect.

(25) See, for example, A. MARTIN: *Nuovo Cimento*, **4**, 369 (1956).

(26) S. GOTÔ: *Prog. Theor. Phys.*, **14**, 417 (1954).

the S-wave scattering and in the anomalous magnetic moment. It would not be too much to say, then, that without consideration of the nucleon « core » the difficulty of the theory would not be got over (*).

The above argument was given in a non-relativistic way, and one may not intend a very generality of such rough estimation of the internal effect of the nucleon core. Even for the low energy events the effect of internal structure, if any, might be essentially of relativistic character. Then, our spacial separation into the « inner » and « outer » parts of the « core » is evidently an oversimplification of the problem. Even so however, it would serve as a rough measure of this interesting problem, and it would give some insight into the reason of the qualitative success of the more specific theories.

As for the question of the anomalous magnetic moment the same situation appears (+). That is, in the P-wave cloud which mainly answers for the meson current contribution to the anomalous magnetic moment, the nucleon structure is irrelevant, while for the nucleon current the core structure seems to be essentially responsible. This problem will be discussed in a forthcoming paper.

(*) The effect of the nucleon recoil was examined in detail in the frame of the intermediate coupling theory ([27]), and the effect of the nucleon pair in the covariant form ([28]). Such corrections could not remedie the defect of the theory.

([27]) H. Hasegawa: private communication.

([28]) S. Machida: *Prog. Theor. Phys.*, **14**, 407 (1955).

(+) The difficulty of the anomalous magnetic moment will not be so simple. The contributions from the strange particles ([29]) would be much more effective than in the S wave scattering, and the strong interaction between pions ([30]) may be also an important factor. Further, as was noted by Yamada, the ambiguity of the charge renormalization will be an additional difficulty of the problem.

([29]) C. Iso and S. Iwao: *Prog. Theor. Phys.*, **16**, 417 (1956).

([30]) W. G. Holladay: *Phys. Rev.*, **101**, 1198 (1956).

RIASSUNTO (*)

Si discutono gli effetti della struttura nucleonica correlando fenomenologicamente col nocciolo nucleonico alcuni fenomeni d'interazione pione-nucleone. Nella regione delle alte energie si interpretano i vari dati sperimentali in termini della struttura nucleonica. I risultati principali sono: 1) Il raggio ottico R del protone valutato dallo scattering elastico π^--p è pressochè costante nel campo d'energia $(1\div5)$ GeV. 2) L'effettivo parametro di collisione della reazione π^--p ha il valore pressochè costante $0.8\cdot10^{-13}$ cm nel campo d'energia $(1\div5)$ GeV. 3) L'energia media dei mesoni prodotti nella reazione π^--p a $(1\div5)$ GeV fanno ritenere la produzione dei mesoni dovuta ad espulsione dal nocciolo nucleonico. 4) Nelle collisioni nucleone-nucleone ad energie dell'ordine di 1 GeV, l'energia media dei mesoni prodotti li fa ritenere espulsi dalla nube mesonica esterna come risulta dal modello isobarico. 5) Si discute la regolarità della riduzione di molteplicità riguardante le stelle di annichilazione protone-antiprotone. Nella regione delle basse energie si imputa la ben nota inapplicabilità della teoria mesonica alle onde S all'effetto della struttura nucleonica interna (Appendice).

(*) *Traduzione a cura della Redazione.*

2049

Zum Aufbau der Quantenfeldtheorie ohne Lagrangeformalismus.

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Summary. — It is investigated to what extent the quantum theory of elementary particles can be built up by using the following principle only: « There exists an irreducible representation of the algebra of creation- and annihilation operators a^+, a, and the infinitesimal transformations of the inhomogeneous Lorentz group can be constructed out of them. » The representation of the inhomogeneous Lorentz group can then be split up in irreducible ones classified by Wigner and describing different sorts of elementary particles. Space-time dependence of all operators is introduced by a unitary transformation of the a^+, a. Energy-momentum-tensors and charge-current vectors are found without use of fields. Fields and field-equations, corresponding to the various sorts of elementary particles, may also be set up but are dispensable. Interactions are introduced as unitary transformations of the a^+, a, mixing the irreducible representations of the Lorentz group. This corresponds to in general non-local interaction terms in the Lagrange formalism.

Einleitung.

Der Lagrangeformalismus ist als Methode zur Gewinnung der Feldgleichungen, zur Quantisierung dieser Felder und zur Aufstellung der physikalisch wichtigen Operatoren die Grundlage der Quantentheorie der Elementarteilchen. Jedoch muß an einigen wichtigen Punkten am Lagrangeformalismus (L.F.) Kritik geübt werden.

1) Obwohl die kanonische Quantisierung der Felder singuläre Vertauschungsrelationen ergibt, die zu divergenten Ausdrücken für physikalische Größen führen, zeigt sich im Rahmen des L.F. keine Möglichkeit zu einer

Abänderung. Dazu wäre es nämlich notwendig, daß die Feldgleichungen nicht mehr als Ausgangspunkt für die Aufstellung von Vertauschungsrelationen zwischen Operatoren dienen.

2) Aus der Lagrangefunktion L lassen sich sowohl im lokalen wie im nichtlokalen Falle Energieimpulstensor $T^{\nu\mu}(x)$ und bei Vorliegen von Eichinvarianzen Stromladungsvektoren $j^{\nu}(x)$ konstruieren, die differentiellen Erhaltungssätzen $T^{\nu\mu}{}_{|\mu}=0$; $j^{\nu}{}_{|\nu}=0$ genügen. (Im nichtlokalen Falle auf dem Umweg über die allgemeine Relativitätstheorie durch Variation der Metrik [1]) $T^{\nu\mu}(x)$ und $j^{\nu}(x)$ sind jedoch nicht eindeutig gegeben. Denn jede Lagrangefunktion L aus der Äquivalenzklasse aller L, die sich nur um eine Divergenz $F^{\varrho}{}_{|\varrho}$ voneinander unterscheiden, führt zwar zu denselben Feldgleichungen, aber nicht zu demselben $T^{\nu\mu}(x)$. Auch die Symmetrisierung des kanonischen Energieimpulstensors und der Übergang zu geordneten Produkten nach der Quantisierung folgt nicht aus dem L.F. Andererseits ist es schwer, die allgemeinste Form von Tensoren anzugeben, welche unter Berücksichtigung der Feldgleichungen Erhaltungssätzen genügen.

3) Beim Übergang vom Heisenbergbild zum Wechselwirkungsbild, welcher im L.F. durch eine unitäre Transformation $U(\sigma)$ bewirkt wird:

$$\text{(E, 1)} \qquad \frac{\hbar}{i}\frac{\delta U(\sigma)}{\delta\sigma(x)}=-H_w(x)\,U(\sigma) \qquad (\sigma=\text{raumartige Fläche})$$

ist trotz der relativistisch kovarianten Schreibweise die Zeittranslation

$$P^4=\int T^{4\nu}(x)\,\mathrm{d}\sigma_\nu=-\int\left(H_0(x)+H\;(x)\right)\mathrm{d}^3x$$

vor den räumlichen infinitesimalen Translationen P^m $(m=1,2,3)$ ausgezeichnet.

Hinsichtlich der erwähnten Punkte bringt unser Formalismus wesentliche Änderungen; denn erstens ist es in ihm zumindest prinzipiell möglich, die kanonischen Vertauschungsrelationen ohne Verlust relativistischer Invarianz abzuändern; zweitens kann zu jedem Feldtyp ein Überblick über die dazu konstruierbaren Vektoren und Tensoren mit Erhaltungssätzen gegeben werden und drittens wird die Auszeichnung der zeitlichen Translation P^4 vor den räumlichen P^m aufgehoben.

Das Ziel dieser Arbeit ist es, die Quantentheorie der Elementarteilchen versuchsweise ohne Verwendung des L.F. neu aufzubauen, und zwar unter

(1) Vgl. WEIDLICH: *Zeits. f. Phys.*, **147**, 288 (1958).

alleiniger Benutzung allgemeiner Voraussetzungen, die nach unserer bisherigen Kenntnis für jede Elementarteilchentheorie zutreffen müssen.

Felder, die entsprechenden Feldgleichungen und Vertauschungsrelationen genügen, lassen sich auch in diesem Rahmen konstruieren, doch spielen sie eine untergeordnete, entbehrliche Rolle. Denn es lassen sich einerseits für jede Teilchensorte alle physikalisch wichtigen Operatoren wie Energieimpulstensor und Stromladungsvektoren ohne Benutzung der zugehörigen Felder konstruieren, andererseits wird auch die Wechselwirkung zwischen den Teilchen ohne direkte Verwendung der Felder eingeführt.

Es läßt sich jedoch zeigen, daß unser Vorgehen dem Auftreten von im allgemeinen nichtlokalen Wechselwirkungsgliedern in der Lagrangefunktion enspricht.

Folgende Prinzipien sind nun der Ausgangspunkt für unsere Betrachtungen:

1) Es soll eine Algebra $\mathcal{A}(a_1, a_2, \ldots)$ von Operatoren a_i existieren (wegen Prinzip 2) mit notwendig unendlichdimensionaler Basis, wie unten gezeigt wird), die einen Hilbertraum $\mathcal{H}$ als irreduciblen Darstellungsraum hat. Per definitionem bildet dann $\mathcal{A}$ in $\mathcal{H}$ ein vollständiges Operatorsystem. Den Vektoren aus $\mathcal{H}$ werden später physikalische Zustände der Elementarteilchen zugeordnet.

2) Aus den Elementen der Algebra $\mathcal{A}$ sollen sich unitäre Operatoren Λ, Γ ($\Lambda^+ = \Lambda^{-1}$; $\Gamma^+ = \Gamma^{-1}$) aufbauen lassen, welche zusammen eine (reducible) unitäre Darstellung der inhomogenen Lorentzgruppe $\mathcal{L}$ in $\mathcal{H}$ bilden. Der Normalteiler $\mathcal{T}$ der Translationen Γ in $\mathcal{L}$ soll dabei nicht auf 1 abgebildet werden, d.h. die Darstellung von $\mathcal{L}$ soll nicht nur eine solche der homogenen Lorentzgruppe $\mathcal{L}/\mathcal{T}$ sein. Eventuell soll $\mathcal{A}$ ausreichen, um eine unitäre Darstellung von $\mathcal{L} \times \mathcal{G}$ in $\mathcal{H}$ aufzubauen, wobei $\mathcal{G}$ eine Tranformationsgruppe für innere Koordinaten ist. Letztere unterscheiden Teilchen eines Multipletts voneinander welche gleiches Verhalten unter $\mathcal{L}$ zeigen (Beispiel: Isospin; $\mathcal{G}$ ist dann eine dreidimensionale Drehgruppe).

3) Die Elemente Λ; Γ der Lorentzgruppe bzw. die infinitesimalen Transformationen $M^{\nu\mu}$; P^ν von $\mathcal{L}$ sind bisher nur abstrakte Operatoren in $\mathcal{H}$. Andererseits sollen sie neben anderen so hergestellten Operatoren $\Omega, \ldots$ physikalische Bedeutung als Impuls-, Drehimpulsoperatoren usw. erhalten und müssen deshalb auch im allgemeinen Raumzeitfunktionen $\Omega(x^\mu)$ werden. Darüber hinaus muß die Anwendung von $\Lambda(a^\nu{}_\mu)$ und $\Gamma(b^\nu)$ auf andere Operatoren $\Omega(x^\mu)$ (Ω sei z.B. ein Skalar) jetzt die Bedeutung raumzeitlicher Drehungen bzw. Translationen haben:

$$\text{(E.2)} \qquad \begin{cases} \Lambda(a^\nu{}_\mu)\, \Omega(x^\mu)\, \Lambda^{-1}(a^\nu{}_\mu) = \Omega(a^\mu{}_\varrho x^\varrho)\,, \\ \Gamma(b^\nu)\, \Omega(x^\mu)\, \Gamma^{-1}(b^\nu) \quad = \Omega(x^\mu + b^\mu)\,, \end{cases}$$

oder in den infinitesimalen Transformationen geschrieben (vgl. (§2)

$$\text{(E, 3)} \qquad \begin{cases} [M^{\nu\mu}(x);\, \Omega(x)] = i\left(x^{\mu}\dfrac{\partial}{\partial x_{\nu}} - x^{\nu}\dfrac{\partial}{\partial x_{\mu}}\right)\Omega(x)\,, \\ [P^{\nu}(x);\, \Omega(x)] \quad = i\dfrac{\partial}{\partial x_{\nu}}\,\Omega(x)\,. \end{cases}$$

Der Übergang zu raumzeitabhängigen Operatoren soll nun dadurch geschehen, daß die Elemente von $\mathcal{A}$ selbst x^{ν}-abhängig angesetzt werden: $\mathcal{A}((a_1(x), a_2(x), \ldots)$.

Dabei dürfen sich die Vertauschungsrelationen aller physikalisch wichtigen Operatoren, z.B. der $M^{\nu\mu}$; P^{ϱ} untereinander nicht ändern. Da sich die Vertauschungsrelationen aller aus den a_i aufgebauten Operatoren aus den algebraischen Relationen zwischen den a_i ergeben, dürfen sich diese Relationen bei der Transformation $a_i \to a_i(x)$ ebenfalls nicht ändern. Weil nun nach Voraussetzung $\mathcal{H}$ ein irreducibler Darstellungsraum von $\mathcal{A}$ ist, kann deshalb $a_i \to a_i(x)$ nur durch eine Äquivalenztransformation bewirkt werden:

$$\text{(E.4)} \qquad a_i(x) = U(x^{\nu})\, a_i\, U^{-1}(x^{\nu})\,.$$

Mit Gl. (E.4) folgt dann für beliebige Ω:

$$\text{(E.5)} \qquad \Omega(x) = U(x^{\nu})\,\Omega\, U^{-1}(x^{\nu})$$

womit die Einführung der Raumzeitabhängigkeit aller Operatoren vollzogen ist. Es wird eine wichtige Aufgabe sein, $U(x)$ zu konstruieren.

4) Im allgemeinen wird die Algebra $\mathcal{A}$ in ein direktes Produkt von Unteralgebren zerfallen:

$$\text{(E.6)} \qquad \mathcal{A} = \prod_{i=1}^{N} \otimes\, \mathcal{A}_i \quad \text{entsprechend} \quad \mathcal{H} = \prod_{i=1}^{N} \times \mathcal{H}_i\,,$$

wo $\mathcal{H}_i$ irreducibler Darstellungsraum von $\mathcal{A}_i$ ist.

Wenn nun die Aufspaltung von $\mathcal{A}$ in die $\mathcal{A}_i$ derart geschieht, daß alle physikalisch wichtigen Operatoren Ω einschließlich der infinitesimalen Transformationen $M^{\nu\mu}$; P^{ν} von $\mathfrak{L}$ die Form haben:

$$\text{(E.7)} \qquad \Omega = \sum_{i=1}^{N} \Omega_i(\mathcal{A}_i)\,,$$

($\Omega_i(\mathcal{A}_i)$ besteht nur aus Elementen von $\mathcal{A}_i$) so zerfällt das physikalische Gesamtsystem in unabhängige Teilsysteme. Liegt etwa speziell ein normierter

Vektor $F = f_1 f_2 \dots f_N \in \mathcal{H}$ $(f_i \in \mathcal{H}_i)$ vor, so wird:

$$\text{(E.8)} \qquad \operatorname{Erw}(\Omega) \equiv (F, \Omega F) = \sum_{i=1}^{N} (f_i, \Omega_i(\mathcal{A}_i) f_i) \quad \text{usw.}$$

Eine solche Separation in unabhängige Teilsysteme liegt z.B. vor, wenn mehrere unabhängige Teilchensorten ohne Wechselwirkung vorhanden sind. Es liegt daher nahe, die Wechselwirkung zwischen diesen zu den $\mathcal{A}_i$ gehörigen, Teilchensorten durch eine Äquivalenztransformation aller $a \in \mathcal{A}$ einzuführen,

$$\text{(E.9)} \qquad a' = S a S^{-1},$$

welche den Zerfall aller Ω in $\sum_{i=1}^{N} \Omega_i(\mathcal{A}_i)$ aufhebt. Zwar zerfällt nun $\Omega' = S\Omega S^{-1}$ in $\sum_{i=1}^{N} \Omega'_i(S\mathcal{A}_i S^{-1})$, aber den $S\mathcal{A}_i S^{-1}$ darf man, wie sich zeigen wird, im allgemeinen nicht mehr eine Teilchensorte zuordnen.

1. – Einführung der Algebra $\mathcal{A}$.

Die infinitesimalen Transformationen von $\mathcal{L}$ werden in § 2) ausführlich behandelt. Wir beachten zunächst nur, daß $\mathcal{A}$ eine unendlichdimensionale Basis haben muß, wenn sich aus $\mathcal{A}$ die hermiteschen Operatoren $M^{\nu\mu}$, P^ν aufbauen lassen sollen. Denn wäre $\mathcal{A}$ endlichdimensional, dann auch jeder irreducible Darstellungsraum $\mathcal{H}$ von $\mathcal{A}$. Andererseits besitzen die vertauschbaren P^ν, falls $\neq 0$, ein kontinuierliches Eigenwertspektrum. (Beweis: P^ν und das unitär transformierte $P'^\nu = \Lambda(a^\varrho{}_\lambda)\, P^\nu \Lambda^{-1}(a^\varrho{}_\lambda) = a^\nu{}_\lambda P^\lambda$ besitzen dasselbe Eigenwertspektrum; also ist mit p^ν auch $p'^\nu = a^\nu{}_\lambda p^\lambda$ ein Eigenwert von P^ν). Dies ist aber nur in einem $\mathcal{H}$ unendlicher Dimension möglich.

Um den Anschluß an den L.F. zu gewinnen, wählen wir für $\mathcal{A}$ eine Menge von Erzeugungs- und Vernichtungsoperatoren $b_n(k)$; $a_m(k)$ die in geeigneter Weise von kontinuierlichen und diskreten Indices k bzw. n abhängen und entweder

$$\text{(1.1)} \qquad (a_n(k) b_m(k') - b_m(k') a_n(k)) \equiv [a_n(k), b_m(k')]_- = \delta_{nm}\, \delta(k - k')$$

oder

$$\text{(1.1*)} \qquad (a_n(k) b_m(k') + b_m(k') a_n(k)) \equiv [a_n(k), b_m(k')]_+ = \delta_{nm}\, \delta(k - k')$$

genügen. (Bezeichnung der Algebren: $\mathcal{A}_-$ bzw. $\mathcal{A}_+$).

Es soll jedoch an anderer Stelle geprüft werden, ob neben dieser Algebra andere gewählt werden können, welche einen neuartigen Einbau innerer Freiheitsgrade der Teilchen erlauben.

Beide Algebren $\mathcal{A}_-$; $\mathcal{A}_+$ besitzen nun kontinuierlich viele inäquivalente irreducible Darstellungen ([2-4]).

Beweis: Bei $\aleph_0 =$ abzählbar unendlich vielen k_i ist der Raum $\mathcal{R}$ der Besetzungszustände $\varphi(n_1 \cdot k_1; n_2 \cdot k_2; \ldots; n_j \cdot k_j, \ldots)$ mit $n_i = 0, 1$ für $\mathcal{A}_+$ und $n_i = 0, 1, \ldots, \infty$ für $\mathcal{A}_-$ bei bekannter Wirkungsweise von $b(k_i)$, $a(k_i)$ ein Darstellungsraum für $\mathcal{A}_+$ bzw. $\mathcal{A}_-$; seine Dimension ist $2^{\aleph_0}$ bzw. $\aleph_0^{\aleph_0}$ also $\aleph_1 =$ kontinuierlich. $\mathcal{R}$ zerfällt in $\aleph_1$ Räume $\mathcal{H}$ der Dimension $\aleph_0$. Jedes $\mathcal{H}$ besteht aus der Äquivalenzklasse aller φ, die sich von einem beliebigen $\varphi_1 \in \mathcal{H}$ nur um endlich viele Besetzungszahlen unterscheiden. Jedes $\mathcal{H}$ ist irreducibler Darstellungsraum von $\mathcal{A}_+$ bzw. $\mathcal{A}_-$. Der Hilbertraum, der den Vakuumzustand enthält, ist ein solches $\mathcal{H}$.

Wir beschränken uns auf die Verwendung des gewohnten Hilbertraumes $\mathcal{H}$, der den Vakuumzustand enthält. (Bezüglich dabei evtl. auftretender Schwierigkeiten vgl. ([3,4])). Zum späteren Gebrauch ist es nützlich, einige aus (1, 1), (1, 1*) folgenden Formeln zusammenzustellen. Wir bezeichnen abkürzend

$a_{n_i}(k_i)$ als $a(\varkappa_i)$; $\delta_{n_i n_j}\delta(k_i - k_j)$ als Δ_{ij}; $\prod_{i=1}^{n} a(\varkappa_i) \equiv (a(\varkappa))^n$ und $(a(\varkappa))^n_{i_1 \ldots i_l}$ ist gleich $(a(\varkappa))^n$ ohne die Faktoren $a(\varkappa_{i_1})$ bis $a(\varkappa_{i_l})$.

Dann gilt für $\mathcal{A}_-$ bzw. bei Gültigkeit von (1, 1) wegen Rekursionsformeln wie:

$$(1.2) \qquad \left[(b(\varkappa))^m, (a(\varkappa))^{n+1}\right] = \left[(b(\varkappa))^m, (a(\varkappa))^n\right] a(\varkappa_{n+1}) - \\ - \sum_{i=1}^{m} \Delta_{n+1,i}\left\{(b(\varkappa))^m_i (a(\varkappa))^n - \left[(b(\varkappa))^m_i, (a(\varkappa))^n\right]\right\}.$$

nach vollständiger Induktion:

$$(1.3) \qquad \left[(a(\varkappa))^n, (b(\varkappa))^m\right] = \sum_{l=1}^{\mathrm{Min}(n,m)} \sum_{(i_1 j_1)\ldots(i_l j_l)} (-1)^{l-1} \Delta_{i_1 j_1} \ldots \Delta_{i_l j_l} (a(\varkappa))^n_{i_1 \ldots i_l} (b(\varkappa))^m_{j_1 \ldots j_l} = \\ = \sum_{l=1}^{\mathrm{Min}(n,m)} \sum_{(i_1 j_1)\ldots(i_l j_l)} \Delta_{i_1 j_1} \ldots \Delta_{i_l j_l} (b(\varkappa))^m_{j_1 \ldots j_l} (a(\varkappa))^n_{i_1 \ldots i_l}.$$

Dabei bedeutet $\sum_{(i_1 j_1)\ldots(i_l j_l)}$ die Summe über alle Arten, $i_1 \ldots i_l$ aus $1, 2, \ldots, n$ und $j_1 \ldots j_l$ aus $1, 2, \ldots m$ herauszugreifen und damit Paare $(i_1 j_1) \ldots (i_l j_l)$ zu bilden. Für $\mathcal{A}_+$ gilt eine ähnliche, jedoch kompliziertere Formel. Zwischen zwei Operatoren

$$(1.4) \qquad \Omega = \underset{\varkappa_1 \varkappa_2}{S}\, b(\varkappa_1)\,\omega(\varkappa_1 \varkappa_2)\, a(\varkappa_2)\,; \qquad \Theta = \underset{\varkappa_3 \varkappa_4}{S}\, b(\varkappa_3)\,\vartheta(\varkappa_3 \varkappa_4)\, a(\varkappa_4)$$

($\underset{\varkappa_1 \varkappa_2}{S}$ bedeutet Summation und Integration über $\varkappa_1$, $\varkappa_2$) gilt sowohl für $\mathcal{A}_-$ wie

([2]) A. S. Wightman und S. S. Schweber: *Phys. Rev.*, **98**, 812 (1955).
([3]) R. Haag: *Kon. Dansk. Mat. Fys. Medd.*, 29, No. 12 (1955).
([4]) S. Albertoni und F. Duimio: *Nuovo Cimento*, **6**, 1192 (1957).

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für $\mathcal{A}_-$:

(1.5) $$[\Omega, \Theta] = \underset{\varkappa_1\varkappa_2\varkappa_3}{S} b(\varkappa_1)\big(\omega(\varkappa_1\varkappa_3)\vartheta(\varkappa_2\varkappa_2) - \vartheta(\varkappa_1\varkappa_3)\omega(\varkappa_3\varkappa_2)\big)\, a(\varkappa_2)$$

und

(1.6) $$[\Omega, a(\varkappa)] = -\underset{\varkappa_2}{S}\, \omega(\varkappa; \varkappa_2)\, a(\varkappa_2)\,; \qquad [\Omega, b(\varkappa)] = +\underset{\varkappa_1}{S}\, b(\varkappa_1)\, \omega(\varkappa_1, \varkappa)\,.$$

Bei speziellen Äquivalenztransformationen S transformieren sich $a(\varkappa)$ und $b(\varkappa)$ kontragredient zueinander. Beweis: Sei

(1.7) $$\left|\begin{aligned} \widetilde{a}(\varkappa) &= S\, a(\varkappa)\, S^{-1} = \underset{\varkappa_1}{S}\, s_a(\varkappa_i\varkappa_1)\, a(\varkappa_1)\,, \\ \widetilde{b}(\varkappa) &= S\, b(\varkappa)\, S^{-1} = \underset{\varkappa_2}{S}\, s_b(\varkappa; \varkappa_2) b(\varkappa_2)\,, \end{aligned}\right.$$

so folgt

(1.8) $$[\widetilde{a}(\varkappa_3); \widetilde{b}(\varkappa_4)] = \varDelta_{34} = \underset{\varkappa_1\varkappa_2}{S}\, s_a(\varkappa_3, \varkappa_1)\, s_b(\varkappa_4\varkappa_2)\, \varDelta_{12}$$

also

$$\underset{\varkappa_1}{S}\, s_a(\varkappa_3; \varkappa_1)\, s_b(\varkappa_4; \varkappa_1) = \varDelta_{34}\,.$$

Daher ist bei diesen Äquivalenztransformationen

(1.9) $$N = \underset{\varkappa}{S}\, b(\varkappa)\, a(\varkappa) = \underset{\varkappa}{S}\, \widetilde{b}(\varkappa)\, \widetilde{a}(\varkappa)$$

eine Invariante. Für Vektoren aus $\mathcal{H}$ der Form $\varphi = b(\varkappa_1) \dots b(\varkappa_n)\varphi_0$ wobei φ_0 durch die Forderung

(1.10) $$a(\varkappa)\varphi_0 = 0$$

definiert ist, besitzt N den Eigenwert n.

In $\mathcal{H}$ können nun $b(\varkappa)$ und $a(\varkappa)$, evtl. nach Durchführung einer kanonischen Transformation = Äquivalenztransformation (*), als hermitesch konjugiert gewählt werden, d.h. $b(\varkappa) = a^+(\varkappa)$. Wenn $b(\varkappa) = a^+(\varkappa)$, so gilt $\Omega = \Omega^+$ in (1.4) dann und nur dann, wenn $\omega^*(\varkappa_1; \varkappa_2) = \omega(\varkappa_2; \varkappa_1)$.

Soll $b(\varkappa) = a^+(\varkappa)$ bleiben, dann dürfen weiterhin nur noch unitäre Transformationen durchgeführt werden.

2. – Konstruktion der infinitesimalen Transformationen der Lorentzgruppe.

Die inhomogene Lorentzgruppe besteht aus den Transformationen

(2.1) $$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu} + b^{\mu}$$

(*) Eine Transformation heißt kanonisch, wenn sie die Vertauschungsrelationen invariant läßt. Bei irreduciblen Darstellungen der Algebra kann dies nur eine Äquivalenztransformation sein.

von vier reellen Koordinaten, wobei

$$(x_1^{\mu} - x_2^{\mu})\, g_{\mu\nu} (x_1^{\nu} - x_2^{\nu}) = (x_1'^{\mu} - x_2'^{\mu})\, g_{\mu\nu} (x_1'^{\nu} - x_2'^{\nu})$$

gelten soll, d.h.

$$a^{\mu}{}_{\nu} a^{\lambda\nu} = g^{\mu\lambda} (\nu, \mu = 1 \ldots 4;\; g_{11} = g_{22} = g_{33} = -g_{44} = 1\,;$$

andere Komponenten $= 0$; per Def. sei $x_{\nu} = g_{\nu\varrho} x^{\varrho}$).

Der infinitesimalen Lorentztransformation $a^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \alpha^{\mu}{}_{\nu}$; $b^{\nu} = \beta^{\nu}$ mit $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$ entspricht bei unitärer Darstellung von $\mathfrak{L}$ der Operator $U = 1 + iK$ mit $K = K^{+}$ und

$$K = \tfrac{1}{2} \alpha_{\mu\nu} M^{\mu\nu} + \beta_{\mu} P^{\mu}\,; \qquad M^{\nu\mu} = -M^{\mu\nu}\,. \tag{2.2}$$

Die hermiteschen infinitesimalen Transformationen $M^{\nu\mu}$, P^{ϱ} genügen den Vertauschungsrelationen:

$$\text{(2.3)} \quad \begin{cases} a) \quad [P^{\mu}, P^{\nu}] = 0\,; \\ b) \quad i[M^{\mu\nu}, P^{\lambda}] = P^{\mu} g^{\nu\lambda} - P^{\nu} g^{\mu\lambda}\,; \\ c) \quad i[M^{\mu\nu}, M^{\varrho\sigma}] = g^{\nu\varrho} M^{\mu\sigma} - g^{\nu\sigma} M^{\mu\varrho} + g^{\mu\sigma} M^{\nu\varrho} - g^{\mu\varrho} M^{\nu\sigma}\,. \end{cases}$$

Der Operator $P = P^{\nu} P_{\nu}$ und der aus dem in je zwei Indices antisymmetrischen $Q_{\nu\varrho\lambda} = P_{\nu} M_{\varrho\lambda} + P_{\lambda} M_{\nu\varrho} + P_{\varrho} M_{\lambda\nu}$ gebildete Operator $Q = \frac{1}{6} Q^{\nu\varrho\lambda} Q_{\nu\varrho\lambda}$ sind mit allen P^{ν}; $M^{\mu\varrho}$ vertauschbar.

Wenn nun $M^{\mu\nu}$, P^{ν} aus den Elementen von $\mathfrak{A}$ aufgebaut werden sollen, so lassen sich aus (2.3c) schon allgemeine Folgerungen ziehen: Sind z.B. alle $M^{\nu\mu}$ Polynome gleichen Grades m in $a(\varkappa)$; $b(\varkappa)$, wobei die Potenzausdrücke m-ten Grades der verschiedenen $M^{\nu\mu}$ untereinander nicht vertauschbar seien, so kann nur $m = 2$ oder $m = \infty$ sein, denn die Vertauschung zweier $M^{\nu\mu}$ gibt dann bei Vorliegen von $\mathfrak{A}_{-}$ in jedem Falle ein Polynom $(2m-2)$-ten Grades in $a(\varkappa)$, $b(\varkappa)$, andererseits soll nach (2.3c) wieder ein $M^{\varrho\lambda}$ entstehen. Aus $(2m-2) = m$ folgt die Behauptung.

Wir wollen uns zunächst auf Systeme freier Elementarteilchen beschränken, die wir folgendermaßen charakterisieren:

1) Die Algebra $\mathfrak{A}$ und der Hilbertraum $\mathfrak{H}$ zerfallen nach (E.6) in Unterstrukturen. Ebenso zerfallen N und $M^{\nu\mu}$, P^{ϱ} nach (E.7) in

$$N = \sum_{i=1}^{N} N_i\,; \qquad M^{\nu\mu} = \sum_{i=1}^{N} M_i^{\nu\mu}\,; \qquad P^{\nu} = \sum_{i=1}^{N} P_i^{\nu}\,. \tag{E.7*}$$

φ_0^i, definiert durch $N\varphi_0^i = 0$, sind die Vakuumszustände von $\mathfrak{H}_i$. Gilt für

ein $\varphi \in \mathcal{H}$: $N_i\varphi = n_i\varphi$, so deuten wir n_i als Anzahl der im Zustand φ vorhandenen Elementarteilchen i-ter Sorte.

2) Es gelte $[P^\nu; N_i] = 0$; $[M^{\nu\mu}; N_i] = 0$. Nach § 3) müssen dann N_i raumzeitlich konstant sein.

3) Die Einteilchenzustände $b(\varkappa)\varphi_0^i$ eines jeden $\mathcal{H}_i$ bilden den Darstellungsraum für eine irreducible unitäre Darstellung von $\mathfrak{L}$. Dadurch ist, wenn wir von inneren Koordinaten einmal absehen, die Art der zugehörigen Elementarteilchen festgelegt. (Die Mehrteilchenzustände gleicher oder verschiedener Elementarteilchen zerfallen im Prinzip wieder in irreducible Räume, welche nach Abtrennung von Schwerpunktanteilen gebundene und freie Zustände beschreiben müßten).

Die irreduciblen unitären Darstellungen von $\mathfrak{L}$ sind nun von WIGNER (6) aufgestellt worden. BARGMANN und WIGNER (7), HAAG (3) u.a. haben bereits die Feldtheorien nach diesem Gesichtspunkt klassifiziert. Unsere Aufgabe ist es, die zugehörigen $M^{\nu\mu}$, P^ϱ aus der Algebra $\mathcal{A}$ zu konstruieren. Dabei wird zugleich der Zusammenhang mit den endlichdimensionalen Darstellungen der homogenen Lorentzgruppe geklärt.

Beachten wir, daß nur jeder Potenzausdruck mit gleichviel Erzeugungs- wie Vernichtungsoperatoren $b_j(\varkappa)$ bzw. $a_j(\varkappa)$ mit N_j vertauschbar ist (sowohl bei $\mathcal{A}_-$ wie $\mathcal{A}_+$), so folgt als einfachste Möglichkeit (Grad $m = 2$):

$$(2.4)\qquad \begin{cases} M_j^{\nu\mu} = \underset{\varkappa_1\varkappa_2}{\mathbf{S}}\, b_j(\varkappa_1)\,\mu_j^{\nu\mu}(\varkappa_1\varkappa_2)\,a_j(\varkappa_2)\,, \\ P_j^\varrho \;\;= \underset{\varkappa_1\varkappa_2}{\mathbf{S}}\, b_j(\varkappa_1)\,\pi_j^\varrho(\varkappa_1\varkappa_2)\,a_j(\varkappa_2)\,. \end{cases}$$

Nach (1.5) sind für die Vertauschungsrelationen der $M_j^{\nu\mu}$, P_j^ϱ diejenigen von $\mu_j^{\nu\mu}(\varkappa_1\varkappa_2)$; $\pi_j^\varrho(\varkappa_1\varkappa_2)$ maßgebend. Man erfüllt daher (2.3) z.B., indem man für $\varkappa$ vier kontinuierliche Variable k^ν wählt,

$$(2.5)\qquad \begin{cases} \mu^{\nu\mu}(k_1{}^\varrho k_2{}^\lambda) = \dfrac{1}{i}\,\delta^4(k_2 - k_1)\left(k_1{}^\nu \dfrac{\partial}{\partial k_{1\mu}} - k_1{}^\mu \dfrac{\partial}{\partial k_{1\nu}}\right) \\ \pi^\varrho(k_1{}^\varrho k_2{}^\lambda) \;= \delta^4(k_2 - k_1)k_1{}^\varrho\,, \end{cases}$$

und

$$(2.6)\qquad [a(k_1{}^\varrho),\, b(k_2{}^\varrho)]_\mp = \delta^4(k_1 - k_2) \qquad \text{setzt.}$$

($\delta^4(k_1 - k_2) =$ vierdimensionale δ-Funktion; Index j fortgelassen).

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(6) E. P. WIGNER: *Ann. Math.*, **40**, 149-204 (1939).
(7) V. BARGMANN and E. P. WIGNER: *Proc. Nat. Ac. Sci.*, **34**, 211 (1948).

Dann entsteht:

$$(2.4^*)\quad \begin{cases} M^{\nu\mu} = \int b(k)\,\frac{1}{i}\left(k^\nu \frac{\partial}{\partial k_\mu} - k^\mu \frac{\partial}{\partial k_\nu}\right) a(k)\,\mathrm{d}^4k\,, \\ P^\varrho \;= \int b(k) k^\varrho a(k)\,\mathrm{d}^4k\,. \end{cases}$$

Wählt man die Basis von $\mathscr{H}_j$ so, daß $b(k) = a^+(k)$, so sind $M^{\nu\mu}$; P^ϱ hermitesch, wie es sein muß.

Irreducible Darstellungen sind nun dadurch gekennzeichnet, daß in ihnen jeder Operator, der mit allen Gruppenelementen vertauschbar ist, wie ein Vielfaches der 1 wirkt. Das trifft zu für $P = P^\nu P_\nu$ und $Q = Q_{\nu\mu\lambda} Q^{\nu\mu\lambda}$. Im Raum der Einteilchenzustände wirkt $Q^{\nu\mu\lambda}$ wie

$$(2.7)\qquad \int b(k)\,\frac{1}{i} \sum_{\substack{\nu\mu\lambda \\ \text{zykl}}} k^\nu \left(k^\mu \frac{\partial}{\partial k_\lambda} - k^\lambda \frac{\partial}{\partial k_\mu}\right) a(k) = 0$$

und P genauso wie

$$\int b(k)\, k^\varrho k_\varrho\, a(k)\,\mathrm{d}^4k\,.$$

Soll dies $= c \cdot 1$ sein, so dürfen nur k^ϱ mit $k^\varrho k_\varrho = -m^2 = \text{const.}$ vorkommen. (Wir betrachten nur reelle $m > 0$).

Um zu einem solchen irreduciblen Einteilchenraum überzegehen, sollen $a(k)$, $b(k)$ derart in $a'(k)$; $b'(k)$ umnormiert werden, daß alle Vertauschungsrelationen zwischen Operatoren erhalten bleiben, wenn man darin d^4k durch $\delta(k_\nu k^\nu + m^2)\,\mathrm{d}^4k$ und a; b durch a'; b' ersetzt. So geht z.B.

$$(2.8)\qquad \Omega = \int b(k)\,\omega(k)\,a(k)\,\mathrm{d}^4k$$

über in

$$\hat{\Omega} = \int b'(k)\,\omega(k)\,a'(k)\,\delta(k_\nu k^\nu + m^2)\,\mathrm{d}^4k = \int \sum_\sigma \frac{1}{2\varepsilon}\, b'(k,\sigma)\,\omega(k,\sigma)\,a'(k,\sigma)\,\mathrm{d}^3\boldsymbol{k}$$

und

$$\Theta = \int b(k_1)\,\vartheta(k_1, k_2)\,a(k_2)\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2$$

in

$$\hat{\Theta} = \int \sum_{\sigma_1\sigma_2} \frac{1}{4\varepsilon_1\varepsilon_2}\, b'(k_1\sigma_1)\,\vartheta(k_1\sigma_1;\, k_2\sigma_2)\,a'(k_2\sigma_2)\,\mathrm{d}^3\boldsymbol{k}_1\,\mathrm{d}^3\boldsymbol{k}_2$$

($\varepsilon = +\sqrt{\boldsymbol{k}^2 + m^2}$; σ unterscheidet das Vorzeichen von $k_4 = \pm\varepsilon$ nach Ausführung der Integration über k_4) Insbesondere gilt der Übergang (2.8) auch für die $M^{\nu\mu}$; P^ϱ vgl. (2.13). Aus (2.8) sieht man, daß die Beschränkung auf einen Unterraum von Einteilchenzuständen mit festem Vorzeichen von k_4 (d.h. physikalisch der Energie) durch Lorentztransformationen nicht zerstört wird.

Die obige Bedingung ist erfüllt, wenn

$$[a'(k,\sigma);\, b'(k_1,\sigma_1)] = \delta_{\sigma\sigma_1}\,\delta^3(\boldsymbol{k}-\boldsymbol{k}_1)\cdot 2\varepsilon \tag{2.9}$$

gesetzt wird. In der Feldtheorie werden statt a', b' häufig

$$a''(k,\sigma) = \frac{a'(k,\sigma)}{\sqrt{2\varepsilon}}\,;\qquad b''(k,\sigma) = \frac{b'(k,\sigma)}{\sqrt{2\varepsilon}} \quad \text{verwendet.} \tag{2.10}$$

Neben den bisher aufgefundenen irreduciblen Darstellungen $\Delta(m, 0)$ von $\mathcal{L}$ mit den Eigenwerten $(-m^2)$ bzw. 0 von P bzw. Q gibt es noch andere $\Delta(m, j)$ mit diskreten Eigenwerten von Q, welche durch die verschiedenen irreduciblen Darstellungen der Wignerschen « little group » (6), im Falle $m > 0$ der dreidimensionalen Drehgruppe, charakterisiert werden. Es liegt nahe, zunächst einmal die Produktdarstellungen von $\Delta(m, 0)$ mit den endlichen irreduciblen Darstellungen $D(r, s)$ der homogenen Lorentzgruppe $\mathcal{L}/\mathcal{T}$ zu bilden und die zugehörigen $M^{\nu\mu}$; P^{ϱ} zu konstruieren.

Bekanntlich ist $\mathcal{L}/\mathcal{T}$ direktes Produkt $\mathcal{G}_1 \times \mathcal{G}_2$ zweier gewöhnlicher Drehgruppen. Sind $D_1(r)$; $D_2(s)$ $(r, s = 0, \frac{1}{2}, 1, \ldots)$ irreducible Darstellungen von $\mathcal{G}_1$ bzw. $\mathcal{G}_2$, dann ist $D_1(r) \times D_2(s) = D(r, s)$ eine solche von $\mathcal{L}/\mathcal{T}$. Die infinitesimalen Transformationen von $\mathcal{L}/\mathcal{G}$ in $D(r, s)$ seien $S^{\nu\mu} = -S^{\mu\nu}$. Jene von $\mathcal{G}_1$; $\mathcal{G}_2$ sind dann:

$$\begin{cases} M_1^1 = S^{23} + iS^{41}\,; & M_1^2 = S^{32} - iS^{41} \\ M_2^1 = S^{31} + iS^{42}\,; & M_2^2 = S^{31} - iS^{42} \\ M_3^1 = S^{12} + iS^{43}\,; & M_3^2 = S^{12} - iS^{43}\,. \end{cases} \tag{2.11}$$

Sind M_i^1; M_i^2 hermitesch, dann ist die Darstellung von $\mathcal{L}/\mathcal{T}$ nicht unitär in $D(r, s)$. Die Gruppe $\mathcal{G}_3$ räumlicher Drehungen mit (S^{23}, S^{31}, S^{12}) besitzt in $D(r, s)$ die irreduciblen Darstellungen: $D_3(r+s) \oplus D_3(r+s-1) \oplus \cdots \oplus D_3(|r-s|)$.

Die $M^{\nu\mu}$; P^{ϱ} für $\Delta(m, 0) \times D(r, s)$ erhält man, indem man von Erzeugungs- und Vernichtungsoperatoren $b_n(k\sigma)$, $a_l(k'\sigma')$ ausgeht $(n, l = 1 \cdots (2r+1)(2s+1) =$ $=$ Dimension von $D(r, s))$, welche

$$[a_l(k\sigma);\, b_n(k'\sigma')]_{\pm} = \delta_{ln}\,\delta_{\sigma\sigma'}\,\delta^3(\boldsymbol{k}-\boldsymbol{k}')\cdot 2\varepsilon \tag{2.12}$$

genügen, und

$$\begin{cases} M^{\nu\mu} = \displaystyle\int \sum_{\sigma n} b_n(k,\sigma)\,\frac{1}{i}\left(k^{\nu}\frac{\partial}{\partial k_{\mu}} - k^{\mu}\frac{\partial}{\partial k_{\nu}}\right) a_n(k,\sigma)\,\frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon} \\ \qquad\qquad + \displaystyle\int \sum_{\sigma n,l} b_n(k,\sigma)\,S_{nl}^{\nu\mu}\,a\,(k,\sigma)\,\frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon}\,; \\ P^{\varrho} = \displaystyle\int \sum_{\sigma n} b_n(k,\sigma)\,k^{\varrho}\,a_n(k,\sigma)\,\frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon}\,. \end{cases} \tag{2.13}$$

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In $M^{\nu 4}$ fällt $\partial/\partial k_4$ weg. $M^{\nu\mu}$; P^ϱ sind zunächst als Operatoren in $\mathscr{H}$ im allgemeinen nicht hermitesch, weil i.a. die Matrix $S^{\nu\mu}{}_{nl}$ nicht hermitesch ist und nicht $b_n = a_n^+$ gilt. Die a, b transformieren sich unter $\mathfrak{L}$ kontragredient zueinander (vgl. (1.6), (1.7)), und zwar $a_n(k, \sigma)$ nach $\Delta(m, 0) \times D(r, s)$; $b_n(k, \sigma)$ nach $(\Delta(m, 0) \times D(r, s))_{\mathrm{adj}}$.

Gl. (1.6) lautet jetzt speziell:

$$(2.14)\quad \begin{cases} [M^{\nu\mu}, a_n(k, \sigma)] = -\dfrac{1}{i}\left(k^\nu \dfrac{\partial}{\partial k_\mu} - k^\mu \dfrac{\partial}{\partial k_\nu}\right) a_n(k\sigma) - S^{\nu\mu}{}_{nl} a(k\sigma) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Summe über } l) \\ [P^\varrho, a_n(k\sigma)] = -k^\varrho a_n(k\sigma)\,; \end{cases}$$

entsprechend für $b_n(k\sigma)$.

Wir wollen nun durch eine kanonische Transformation der $a_n(k\sigma)$, $b_m(k\sigma)$ mit k^ν-abhängigen Koeffizienten bewirken, daß $\Delta(m, 0) \times D(r, s)$ in irreducible $\Delta(m, j)$ aufspaltet. Gleichzeitig können dabei die $M^{\nu\mu}$; P^ϱ hermitesch gemacht werden, da die $\Delta(m, j)$ als unitäre Darstellungen gewählt werden können.

In Produkten $k_\nu \cdot a_n(k\sigma)$ verhält sich nun k_ν wie ein Vektor, d.h. nach $D(\frac{1}{2}, \frac{1}{2})$, sodaß $k_\nu a_n(k\sigma)$ sich nach $\Delta(m, 0) \times D(r, s) \times D(\frac{1}{2}, \frac{1}{2})$ transformiert. Genauso kann man den Raum aller Produkte von homogenen Polynomen M-ten Grades in den k_ν mit den $a_n(k\sigma)$ betrachten, wobei $M = 2\,\mathrm{Min}\,(r, s)$ sein soll. Dieser Raum transformiert sich nach $\Delta(m, 0) \times D(r, s) \times D^M(\frac{1}{2}, \frac{1}{2})$. Bei der Ausreduktion von $D(r, s) \times D^M(\frac{1}{2}, \frac{1}{2})$ treten auch die Darstellungsräume $D(r+s, 0) \oplus \oplus D(r+s-1, 0) \cdots \oplus D(|r-s|, 0)$ auf, wie man sich leicht durch wiederholte Anwendung der Formel

$$D(r, s) \times D(p, q) = \sum_{R=|r-p|}^{r+p} \sum_{S=|s-q|}^{s+q} \oplus D(R, S)$$

überzeugt. Die Dimension der Räume $D(r+s, 0) \cdots D(|r-s|, 0)$ ist gerade $(2r+1)(2s+1)$, d.h. gleich der Anzahl der $a_n(k, \sigma)$ bzw. $b_n(k, \sigma)$ bei festen k, σ. Man kann daher von den $a_n(k\sigma)$; $b_n(k\sigma)$ durch lineare, k_ν-abhängige Transformationen zu solchen $\tilde{a}_n(k\sigma)$, $\tilde{b}_l(k\sigma)$ übergehen, welche die Räume $\Delta(m, 0) \times \times D(r+s, 0) \cdots \Delta(m, 0) \times D(|r-s|, 0)$ aufspannen. Es zeigt sich nun, daß damit schon die Ausreduktion von $\Delta(m, 0) \times D(r, s)$ stattfand. Da Ausreduktionen immer durch Äquivalenztransformationen gewonnen werden, ist der Übergang a_n, $b_l \to \tilde{a}_n$, $\tilde{b}_l$ eine kanonische Transformation, die auch den Eigenwert $(-m^2)$ von P nicht ändert. Zu beweisen bleibt:

$$(2.15)\quad \Delta(m, 0) \times D(r, s) = \Delta(m, r+s) \oplus \Delta(m, r+s-1) \oplus \cdots \oplus \Delta(m, |r-s|)$$

wobei also $\Delta(m, j) \equiv \Delta(m, 0) \times D(j, 0)$ irreducible Darstellungen von $\mathfrak{L}$ sein sollen. Dazu beachten wir, daß $Q^{\nu\mu\lambda}$ bei Einteilchenzuständen nach Trans-

formation der a, b in $\widetilde{a}$, $\widetilde{b}$ wie

$$\int \sum_{\substack{\nu\mu\lambda \\ \text{zykl}}} \sum_{\sigma n l} \widetilde{b}_n(k\sigma) k^\nu \widetilde{S}_{nl}^{\mu\lambda} \widetilde{a}\,(k\sigma) \frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon}$$

wirkt (der andere Anteil fällt nach (2.7) weg), wobei in $D(j, 0)$ wegen $M_i^2 = 0$ nach (2.11) gilt:

$$\widetilde{S}^{23} = i\widetilde{S}^{41}\,; \qquad \widetilde{S}^{31} = i\widetilde{S}^{42}\,; \qquad \widetilde{S}^{12} = i\widetilde{S}^{43}\,,$$

und

$$M_1^1 = 2\widetilde{S}^{23}\,; \qquad M_2^1 = 2\widetilde{S}^{31}\,; \qquad M_3^1 = \widetilde{S}2^{12}\,.$$

Damit wird schließlich

$$Q = \int \sum_{\sigma n l} \widetilde{b}_n(k\sigma)(-\,m^2)\,\boldsymbol{M}_{nl}^2 \widetilde{a}_l(k\sigma) \frac{\mathrm{d}^3 k}{2\varepsilon}$$

mit $\boldsymbol{M}^2 = \sum_i (M_i^1)^2$. In jedem $\Delta(m, j)$ wirkt daher Q wie $(-\,m^2)j(j+1)$, womit die Irreducibilität von $\Delta(m, j)$ gezeigt ist.

Am Beispiel von «Vektorteilchen», d.h. $\Delta(m, 0) \times D(\frac{1}{2}, \frac{1}{2})$ wollen wir den Übergang $a_\nu(k\sigma)$; $b^\nu(k\sigma) \rightarrow \widetilde{a}_\nu(k\sigma)$; $\widetilde{b}^\nu(k\sigma)$ erörtern: Der Raum der $k_\varrho a_\nu(k\sigma)$ bzw. $k_\varrho b_\nu(k\sigma)$ enthält die irreduciblen Teilräume $D(0, 0)$ und $D(1, 0)$ mit den zugehörigen Basen:

$D(0,0)$: $$\widetilde{a}_0(k\sigma) = \frac{1}{m} a_\nu(k\sigma) k^\nu; \qquad \widetilde{b}^0(k\sigma) = -\frac{1}{m} b^\mu(k\sigma) k_\mu\,,$$

(2.16) $$\widetilde{a}_{\,}(k\sigma) = \frac{1}{m}\{k^q a_{\,} - k^r a_{\,} + k^4 a_{\,} - k^p a_4\}\,,$$

$D(1,0)$: $$\widetilde{b}^p(k\sigma) = -\frac{1}{m}\{k_{\,} b^r - k_{\,} b^q + k_4 b^p - k_{\,} b^4\}\,,$$

$(p, q, r = 1, 2, 3$ zyklisch$)$.

Man bestätigt leicht die kanonischen Vertauschungsrelationen für die $\widetilde{a}_\nu$; $\widetilde{b}^\nu$. Indem man die a, b durch die $\widetilde{a}$, $\widetilde{b}$ ausdrückt, kann man alle Operatoren, insbesondere die $M^{\nu\mu}$, P^ϱ nach (2.13) auf die irreduciblen Teilräume beziehen. $M^{\nu\mu}$, P^ϱ müssen dann in $\sum_{j=|r-s|}^{r+s} M_j^{\nu\mu}$; $\sum_{j=|r-s|}^{r+s} P_j^\varrho$ zerfallen, wie es (2.15) und (E.7*) verlangen. Sie werden hermitesch, wenn wir nunmehr $b^q = a_q^+$ fordern. Dem darstellungstheoretischen Zerfall (2.15) entspricht die Tatsache, daß höhere Spinorfelder, um nur eine Teilchensorte zu beschreiben, neben der Feldgleichung

noch bestimmten Nebenbedingungen nach der Art der Lorentzkonvention genügen müssen.

Nach der Ausreduktion kann man natürlich innerhalb einer zu $\Delta(m, j)$ gehörigen Algebra $\mathcal{A}_{mj}$ durch eine unitäre Transformation von den $a_n(k\sigma)$, $b^n(k\sigma) = a^+(k\sigma)$ zu anderen Vernichtungs- und Erzeugungsoperatoren $c_n(k\sigma)$ $d^n(k\sigma) = c_n^+(k\sigma)$ übergehen. Faßt man die a_n, c_n in Spalten $\boldsymbol{a}(k\sigma)$, $\boldsymbol{c}(k\sigma)$ zusammen, so laute der Übergang (ohne σ)

$$\boldsymbol{c}(\varkappa) = \int W(\varkappa, k)\, \boldsymbol{a}(k) \frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon} \tag{2.17}$$

mit unitärer Matrix $W(\varkappa, k)$. Endliche Lorentztransformationen $\Lambda(a^\nu{}_\mu)$; $\Gamma(b^\varrho)$ wirken nun auf $\boldsymbol{a}(k^\mu)$ folgendermaßen, wie aus (2.14) hervorgeht:

$$\left\{\begin{aligned} \boldsymbol{a}'(k) &= \Lambda(a^\nu{}_\mu)\, \boldsymbol{a}(k)\, \Lambda^{-1}(a^\nu{}_\mu) = L(k'^\mu)\, \boldsymbol{a}(k'^\mu) \\ \boldsymbol{a}''(k) &= \Gamma(b^\varrho)\, \boldsymbol{a}(k)\, \Gamma^{-1}(b^\varrho) = \boldsymbol{a}(k''^\mu)\,, \end{aligned}\right. \tag{2.18}$$

mit $L(k'^\mu)$ als Transformationsmatrix, $k'^\mu = a^\mu{}_\nu k^\nu$; $k''^\mu = \exp[-ib^\varrho k_\varrho]k^\mu$. Damit ist das Transformationsverhalten von $\boldsymbol{c}(\varkappa)$ bereits festgelegt durch:

$$\boldsymbol{c}'(\varkappa) = \Lambda \boldsymbol{c}(\varkappa)\Lambda^{-1} = \int W(\varkappa, k)\Lambda \boldsymbol{a}(k)\Lambda^{-1} \frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon} = \int W(\varkappa, k)\, L(k')\, \boldsymbol{a}(k') \frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon}\,. \tag{2.19}$$

Hier kann $\varkappa$ evtl. noch ein diskreter Index sein. Verlangt man zusätzlich, daß die $\boldsymbol{c}(k_1)$ sich wie $\boldsymbol{a}(k)$ nach (2.18) transformieren sollen, so muß gelten:

$$\begin{aligned} \int W(k_1 k_2)\, L(k_2')\, \boldsymbol{a}(k_2') \frac{\mathrm{d}^3\boldsymbol{k}_2}{2\varepsilon_2} = L(k_1') \int W(k_1' k_2)\, \boldsymbol{a}(k_2) \frac{\mathrm{d}^3\boldsymbol{k}_2}{2\varepsilon_2} = \\ = \int L(k_1')\, W(k_1' k_2')\, \boldsymbol{a}(k_2') \frac{\mathrm{d}^3\boldsymbol{k}_2}{2\varepsilon_2}\,. \end{aligned} \tag{2.20}$$

(letzteres durch Substitution $k_2 \to k_2'$ mit $\mathrm{d}^3\boldsymbol{k}_2/2\varepsilon_2 = \mathrm{d}^3\boldsymbol{k}_2'/2\varepsilon_2'$). Durch Vergleich folgt die Bedingung für $W(k_1 k_2)$:

$$L(k_1')\, W(k_1' k_2')\, L^{-1}(k_2') = W(k_1 k_2)\,. \tag{2.21}$$

Die Existenz solcher $W(k_1 k_2)$ wird bei der Aufstellung von Diracfeldern usw in §§ 3, 4 nützlich sein.

3. – Die Raumzeitabhängigkeit der Operatoren.

In der Einleitung wurde gezeigt, daß die Raumzeitabhängigkeit aller Operatoren nur durch eine Äquivalenztransformation der a, a^+ eingeführt werden kann, die überdies unitär sein muß, wenn hermitesche Operatoren hermitesch

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bleiben sollen. Nach (E.4) setzen wir also $a_n(x^\nu) = U(x^\nu)\, a_n U^+(x^\nu)$ und fordern nach (E.3) und (2.14)

$$U(x^\nu)\,[P^\varrho, a_n(k\sigma)]\; U^+(x^\nu) \equiv -\, U(x^\nu)\, k^\varrho\, a_n(k\sigma)\; U^+(x^\nu) = = i(\partial/\partial x_\varrho)\; U(x^\nu)\, a_n(k\sigma)\, U^+(x^\nu)\,. \tag{3.1}$$

Diese Gleichung wird allgemein gelöst durch die Bedingung

$$\begin{cases} i(\partial/\partial x_\nu)\, U(x^\mu) = U(x^\mu) P^\nu\,, \\ -\, i(\partial/\partial x_\nu)\, U^+(x^\mu) = P^\nu\, U^+(x^\mu)\,. \end{cases} \tag{3.2}$$

Da die P^ν untereinander vertauschbar sind, läßt sich (3.2) formal lösen mit

$$U(x^\mu) = \exp\,[-\, i x_\nu P^\nu]\,; \qquad U^+(x^\mu) = \exp\,[i x_\nu P^\nu]\,. \tag{3.3}$$

Die P^ν sind gegeben durch (2.13). Es wird nun nach (3.1) insbesondere:

$$a_n(x^\nu, k\sigma) = \exp\,[i k_\varrho x^\varrho]\, a_n(k\sigma)\,. \tag{3.4}$$

Gleichung (3.2) ist die kovariant verallgemeinerte Schrödingergleichung ohne Auszeichnung von P^4.

Wenn die Operatoren in der oben durchgeführten Weise mit Raumzeitabhängigkeit ausgestattet sind, nennen wir dies in unserem Formalismus Heisenbergbild. Die Zustandsvektoren sind dann x^ν-unabhängig. Es bereitet aber hier keine Schwierigkeiten, zum Schrödingerbild überzugehen, indem man die Zustände von $\mathcal{H}$ durch

$$\varphi(x^\nu) = U^+(x^\nu)\varphi \tag{3.5}$$

voll raumzeitabhängig macht, während nun sämtliche Operatoren raumzeitunabhängig bleiben. (Die Erwartungswerte sind in beiden Bildern dieselben).

Anstatt in allen Operatoren die a durch $a(x^\varrho)$ zu ersetzen, kann deren Raumzeitabhängigkeit manchmal direkt nach

$$[P^\nu(x), \Omega(x)] = i(\partial/\partial x_\nu)\, \Omega(x) \tag{E.3}$$

bestimmt werden; z.B. folgt wegen $[P^\nu(x), P^\mu(x)] = 0$ sofort:

$$P^\nu(x) = P^\nu\,, \qquad \text{d.h. } P^\nu \text{ ist } x^\nu\text{-unabhängig.} \tag{3.6}$$

Weiter folgt aus

$$[P^2,\ M^{23}(x)] = i(\partial/\partial x_2)\, M^{23}(x) = -iP^3\,,$$

$$[P^3,\ M^{23}(x)] = i(\partial/\partial x_3)\, M^{23}(x) = +iP^2\,,$$

$$(\partial/\partial x_1)\, M^{23} = (\partial/\partial x_4)\, M^{23} = 0\,,$$

unmittelbar:

$$M^{23}(x) = M^{23}(0) - x^2 P^3 + x^3 P^2\,. \tag{3.7}$$

Die anschauliche Bedeutung dieser Gleichung zwischen Impulsen und Drehimpulsen ist offenbar.

Unter freien Feldern $\psi(x)$ verstehen wir nun allgemein Operatoren, die linear in den Erzeugungs- bzw. Vernichtungsoperatoren sind. Aus (3.4) folgt, daß Felder, die aus den zu einer Darstellung $\Delta(m, j)$ gehörigen $a_n(x^\nu, k\sigma)$ aufgebaut sind, stets der Klein-Gordongleichung

$$(\square - m^2)\, \psi(x) = 0 \tag{3.8}$$

genügen.

Aber auch Felder, die einer Feldgleichung erster Ordnung genügen, lassen sich in bestimmten Fällen konstruieren. Wir betrachten z.B. die Darstellung $\Delta(m, 0) \times D((\frac{1}{2}\,0) \oplus D(0\,\frac{1}{2}))$ der Dirakspinoren. Neben dem Spinorfeld $\psi_1(x) = \boldsymbol{a}(x, k)$ lassen sich noch andere Spinoren $\psi_2(x) = \boldsymbol{c}(x, k)$ finden, wenn es Matrizen $W(k_1 k_2)$ gibt, die (2.21) genügen. Dies ist in der Tat für

$$W(k_1 k_2) = 2\varepsilon_1\, \delta^3(\boldsymbol{k}_1 - \boldsymbol{k}_2)\gamma_\nu k_1^\nu \tag{3.9}$$

der Fall, (γ_ν = Diracmatrizen; $L(k)$ ist hier die k-unabhängige Spinortransformation). Es ist dann $\psi_2(x) = k^\nu \gamma_\nu \boldsymbol{a}(x, k)$. Die Linearkombination

$$\psi(x) = \int (\gamma_\nu k^\nu + im)\, \boldsymbol{a}(x, k)\, \frac{\mathrm{d}^3\boldsymbol{k}}{2\varepsilon} \tag{3.10}$$

genügt offenbar der freien Diracgleichung

$$(\gamma_\mu\, (\partial/\partial x_\mu) + m)\, \psi(x) = 0\,. \tag{3.11}$$

4. – Konstruktion physikalisch wichtiger Operatoren.

Die Felder haben bei der bisherigen Betrachtung nur eine untergeordnete Rolle gespielt. Auch bei der Konstruktion der möglichen Energieimpuls-

tensoren und Stromladungsvektoren für verschiedene Teilchensorten werden sie nicht benötigt. Wir stellen für Energieimpulstensor $T^{\nu\mu}$ und Stromladungsvektor j^ν folgende Forderungen:

1) Transformation als symmetrischer Tensor $T^{\nu\mu}$ bzw. Vektor j^ν unter Lorentztransformationen.

2) $[P^\mu, j_\mu] = 0$; $[P^\mu, T_{\nu\mu}] = 0$ (über μ summiert). Dies sind nach Übergang zur x^ϱ-Abhängigkeit die Erhaltungssätze.

3) $T^{\nu\mu}$, j_ν können als bilinear in Erzeugungs- und Vernichtungsoperatoren angenommen werden. In diesem Falle sind sie vertauschbar mit N, d.h. gleichzeitig mit der Teilchenzahl meßbar (diese Bedingung kann fallengelassen werden).

4) Nach Übergang zu $T^{\nu\mu}(x)$ gilt: $P^\mu = \int T^{\mu 4}(x)\,\mathrm{d}^3x$.

Um 1) und 3) zu erfüllen, machen wir folgenden Ansatz:

$$(4.1)\qquad \left\{\begin{aligned} T^{\nu\mu} &= \int \boldsymbol{a}^{+T}(k_1)\,\omega^{\nu\mu}(k_1k_2)\,\boldsymbol{a}(k_2)\,\mathrm{d}\tau\,, \\ j^\nu &= \int \boldsymbol{a}^{+T}(k_1)\,\vartheta^\nu(k_1k_2)\,\boldsymbol{a}(k_2)\,\mathrm{d}\tau\,. \end{aligned}\right.$$

Dabei ist $\boldsymbol{a}^{+T}(k)$ transponiert zu $\boldsymbol{a}^+(k)$; $\mathrm{d}\tau = \mathrm{d}^3\boldsymbol{k}_1\,\mathrm{d}^3\boldsymbol{k}_2/4\varepsilon_1\varepsilon_2$. Summen über $\sigma = sgn\,(k_4)$ wurden weggelassen.

Soll gelten

$$(4.2)\qquad \left\{\begin{aligned} \Lambda(a^\lambda{}_\varrho)\,T^{\nu\mu}\Lambda^{-1}(a^\lambda{}_\varrho) &= a^\nu{}_\varrho a^\mu{}_\lambda T^{\varrho\lambda}\,, \\ \Lambda(a^\lambda{}_\varrho)\,j^\mu\Lambda^{-1}(a^\lambda{}_\varrho) &= a^\mu{}_\varrho j^\varrho\,, \end{aligned}\right.$$

so müssen wegen (2.18), ergänzt durch $\boldsymbol{a}^{+T'}(k) = \boldsymbol{a}^{+T}(k')L^{-1}(k')$, $\omega^{\nu\mu}$; ϑ^ν die Gleichungen:

$$(4.3)\qquad \left\{\begin{aligned} L(k_1')\,\omega^{\sigma\lambda}(k_1'k_2')\,L^{-1}(k_2') &= a^\varrho{}_\nu a^\lambda{}_\mu \omega^{\nu\mu}(k_1\,k_2)\,, \\ L(k_1')\,\vartheta^\varrho(k_1'k_2')\,L^{-1}(k_2') &= a^\varrho{}_\nu \vartheta^\nu(k_1\,k_2)\,, \end{aligned}\right.$$

mit $k'^\nu = a^\nu{}_\mu k^\mu$ erfüllen (vgl. auch (2.21)). Forderung 2) bedeutet dann, wenn man Gl. (2.14) benutzt:

$$(4.4)\qquad (k_1{}^\mu - k_2{}^\mu)\,\omega_{\nu\mu}(k_1k_2) \doteq 0\,; \qquad (k_1{}^\mu - k_2{}^\mu)\,\vartheta_\mu(k_1k_2) \doteq 0\,.$$

Es ist nun nicht schwer, im Falle skalarer oder Diracspinorteilchen $T^{\nu\mu}$ und j^ϱ zu finden, die allen Bedingungen genügen.

a) Skalare Teilchen:

$L(k)$ ist hier 1. Wir führen neben k_1^ν; k_2^ν noch ein:

$$(4.5)\qquad \begin{cases} k^\nu = k_1^\nu - k_2^\nu\,; \qquad K^\nu = k_1^\nu + k_2^\nu\,; \qquad k_\nu k^\nu = \varkappa\,; \\ k_1^\nu k_{2\nu} = -\,m^2 - \dfrac{\varkappa}{2}\,;\;\; K^\nu K_\nu = -\,4m^2 - \varkappa\,; \qquad K^\nu k_\nu = 0\,. \end{cases}$$

Der allgemeinste Ansatz, der wegen (4.3) für $\omega^{\varrho\lambda}(k_1k_2)$, $\vartheta^\nu(k_1k_2)$ in Frage kommt, ist:

$$(4.6)\qquad \begin{cases} \omega^{\varrho\lambda}(k_1k_2) = f_1(\varkappa)k_1^\varrho k_1^\lambda + f_2(\varkappa)k_2^\varrho k_2^\lambda + f_3(\varkappa)(k_1^\varrho k_2^\lambda + k_2^\varrho k_1^\lambda) + f_4(\varkappa)g^{\varrho\lambda} \\ \vartheta^\nu(k_1k_2) \;\;= h_1(\varkappa)k_1^\nu + h_2(\varkappa)k_2^\nu\,. \end{cases}$$

Aus (4.4) folgt dann:

$$(4.7)\qquad \begin{cases} f_1(\varkappa) = f_2(\varkappa) = f(\varkappa)\,; \qquad f_3(\varkappa) = g(\varkappa)\,; \qquad f_4(\varkappa) = (\varkappa/2)\,(g(\varkappa) - f(\varkappa))\,, \\ h_1(\varkappa) = h_2(\varkappa) = h(\varkappa)\,. \end{cases}$$

Mögliche Energieimpulstensoren und Materiestromvektoren sind also bei skalaren Teilchen

$$(4.8)\qquad T^{\nu\mu}(x) = \int a^+(k_1)\Big\{f(\varkappa)\,(k_1^\nu k_1^\mu + k_2^\nu k_2^\mu) + g(\varkappa)(k_1^\nu k_2^\mu + k_1^\mu k_2^\nu) + \\ + \frac{\varkappa}{2}\,(g(\varkappa) - f(\varkappa))\,g^{\nu\mu}\Big\}\exp\,[ix_\varrho\,(k_2^\varrho - k_1^\varrho)]\,a(k_2)\,\mathrm{d}\tau\,,$$

$$(4.9)\qquad j^\nu(x) = \int a^+(k_1)\,h(\varkappa)(k_1^\nu + k_2^\nu)\exp\,[ix_\varrho(k_2^\varrho - k_1^\varrho)]\,a(k_2)\,\mathrm{d}\tau\,.$$

Im Vergleich hierzu besitzt der aus dem Lagrangeformalismus für reelle skalare Felder folgende kanonische Tensor $T^{\nu\mu}(x)$ auch Anteile mit zwei Erzeugungs- oder zwei Vernichtungsoperatoren, die also nicht mit dem Anzahloperator N vertauschbar sind. Für jeden Anteil gilt eine (4.4) entsprechende Gleichung. Der in a, a^+ bilineare Teil stimmt mit unserem für $f(\varkappa) = 0$; $g(\varkappa) = 1$ überein. Die Forderung 4) wird erfüllt, wenn $(f(0) + g(0)) = 1$ ist. Ein Ausdruck für $j^\nu(x)$ kann bekanntlich im Lagrangeformalismus nicht in einfacher Weise erhalten werden, weil bei reellen Feldern keine Eichinvarianz von L vorliegt.

b) Diracteilchen:

Hier sind die Möglichkeiten zur Bildung von $\omega^{\varrho\lambda}$; ϑ^ν erheblich größer. Zum Beispiel sind

$$(4.10)\qquad \omega^{\nu\mu}(k_1k_2) = q(\varkappa)\omega_1^{\nu\mu} + p(\varkappa)\omega_2^{\nu\mu}$$

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mit

$$\omega_1^{\nu\mu} = \gamma^\nu K^\mu + \gamma^\mu K^\nu \,; \qquad \omega_2^{\nu\mu} = \gamma^\varrho k_\varrho (\gamma^\nu K^\mu + \gamma^\mu K^\nu)\gamma^\lambda k_\lambda$$

und

$$\vartheta^\nu(k_1 k_2) = u(\varkappa)\gamma^\nu + v(\varkappa)\gamma^\varrho k_\varrho \gamma^\nu \gamma^\lambda k_\lambda \,. \tag{4.11}$$

Ausdrücke, die (4.3) genügen, wobei L wieder die Spinortransformation ist. Man sieht leicht, daß auch die Erhaltungssätze (4.4) erfüllt sind, wenn

$$q(\varkappa) + \varkappa p(\varkappa) = 0 \,; \qquad u(\varkappa) + \varkappa v(\varkappa) = 0 \tag{4.12}$$

gilt. $T^{\nu\mu}$, j^ϱ lassen sich wieder nach (4.1) hinschreiben.

5. – Einführung der Wechselwirkung.

Wir wollen die Wechselwirkung unter der einschränkenden Voraussetzung einführen, daß die Operatoren des wechselwirkenden Systems S' in demselben Hilbertraum $\mathcal{H}$ wirken sollen wie die Operatoren des zugehörigen freien Systems S. Alle physikalischen Größen von S' sollen ferner innerhalb gewisser Grenzen stetig von einer oder mehreren Kopplungskonstanten g abhängen. Diese Forderungen sind erfüllt, wenn sich S' von S nur dadurch unterscheidet, daß in ihm anstelle von P_ν; $M_{\nu\mu}$:

$$\widetilde{P}_\nu = VP_\nu V^{-1}; \qquad \widetilde{M}_{\nu\mu} = VM_{\nu\mu}V^{-1} \,, \tag{5.1}$$

als die physikalischen Impuls- und Drehimpulsoperatoren gedeutet werden, wobei

$$V = \exp[igT]; \qquad T = T^+ \,, \tag{5.2}$$

ein noch zu wählender unitärer Wechselwirkungsoperator ist. Die x-Abhängigkeit aller Operatoren Ω wird nun in S' durch die Transformation

$$\Omega'(x) = U'(x)\Omega U'^+(x) \quad \text{mit} \quad U'(x) = \exp[-ix_\nu \widetilde{P}^\nu] \,, \tag{5.3}$$

vermittelt (vgl. (3.3)). Daraus folgt speziell:

$$\partial/\partial x_\mu \widetilde{P}'_\nu(x) = 0 \,. \tag{5.4}$$

Daher kann V nur in trivialer Weise von x_μ abhängen:

$$V = V_0 \cdot F(x) \qquad \text{mit} \qquad [F(x), P^\nu] = 0 \,,$$

und soll deshalb von vornherein als von x_μ unabhängig vorausgesetzt werden. Der Zusammenhang zwischen den $\Omega'(x)$ und den ungestörten

$$\Omega(x) = U(x)\Omega U^{-1}(x)\,,$$

lautet:

$$\Omega'(x) = W(x)\Omega(x)W^{-1}(x)\,, \tag{5.5}$$

mit

$$W(x) = U'(x)U^{-1}(x)\,. \tag{5.6}$$

Allgemein gilt:

$$\tilde{P}_\nu = VP_\nu V^{-1} = P_\nu + g\pi_\nu\,. \tag{5.7}$$

Wie man leicht sieht, genügt der Operator $W(x)$ der Gleichung

$$i\frac{\partial W}{\partial x_\nu} = g\pi^{\nu'}(x)W(x) \qquad \text{mit} \qquad \pi^{\nu'}(x) = U'(x)\pi U'^{-1}(x)\,, \tag{5.8}$$

oder

$$i\frac{\partial W^{-1}(x)}{\partial x_\nu} = g\pi^\nu(x)W^{-1}(x) \qquad \text{mit} \qquad \pi^\nu(x) = U(x)\pi^\nu U^{-1}(x)\,. \tag{5.8'}$$

$W^{-1}(x)$ entspricht in der üblichen Formulierung dem Operator, der den übergang von Feldern des Heisenbergbildes in die freien Felder des Wechselwirkungsbildes vermittelt.

Wenn die Impulse und Drehimpulse von S' durch eine Äquivalenztransformation (5.1) entstehen, so unterscheidet sich andererseits S' insofern mathematisch nicht von S, als die Operatoren

$$\tilde{\Omega}'(x) = U'(x)\tilde{\Omega}U'^{-1}(x)\,; \qquad \tilde{\Omega} = V\Omega V^{-1}\,, \tag{5.9}$$

aus S' dieselbe Raumzeitabhängigkeit zeigen wie die

$$\Omega(x) = U(x)\text{—}U^{-1}(x) \qquad \text{aus} \qquad S\,.$$

Beweis: Es ist $\tilde{\Omega}'(x) = V\Omega(x)V^{-1}$. Besteht nun ein System von raumzeitlichen Differentialgleichungen $D_j(x_1\Omega_1(x), \Omega_2(x), \dots) = 0$ zwischen Operatoren $\Omega_n(x)$ aus S, so gilt wegen der x-Unabhängigkeit von V auch:

$$VD_j(x, \Omega_1(x), \Omega_2(x), \dots)V^{-1} = D_j(x, \tilde{\Omega}_1'(x), \tilde{\Omega}_2'(x), \dots) = 0\,.$$

([8]) D. Hall and A. Wightman: *Dan. Mat. Fys. Med.*, **31**, n. 5 (1957).

Wegen dieser mathematischen Äquivalenz von S and S' (vgl. in diesem Zusammenhang HALL and WIGHTMAN [8]) kann der physikalische Unterschied zwischen S und S' nur dadurch zustande kommen, daß die mit einer bestimmten Basis von Erzeugungsoperatoren $a^+(\varkappa)$ durch Anwenden auf ein Vakuum φ_0 aufgebauten Zustände von $\mathcal{H}$ wie üblich als Ein- bzw. Mehrteilchenzustände gedeutet werden, während dies für die analog durch Anwenden von $\tilde{a}^+(\varkappa) = Va^+(\varkappa)V^{-1}$ auf $V\varphi_0$ erzeugten Zustände nicht mehr gilt. Allgemein spielen also die Operatoren $\tilde{\Omega}'(x)$ in S' abgesehen von $\tilde{P}'_\nu(x)$ und $\tilde{M}'_{\nu\mu}(x)$ physikalisch nicht dieselbe Rolle wie die $\Omega(x)$ in S.

Der Operator T in (5.2) kann aus den $a(\varkappa)$, $a^+(\varkappa)$ aufgebaut werden, da diese ein vollständiges Operatorsystem in $\mathcal{H}$ bilden. Aus der Formel

$$\tilde{\Omega} = \exp[igT]\Omega\exp[-igT] = \sum_{n=0}^{\infty}\frac{(ig)^n}{n!}[T, \Omega]_n, \tag{5.10}$$

$$\text{mit } [T, \Omega]_n = \big[T, \dots [T[T, \Omega]] \dots\big] \quad (n \text{ Klammern})$$

folgt leicht, daß $\tilde{a}(\varkappa)$, $\tilde{a}^+(\varkappa)$ entweder Linearkombinationen oder Potenzausdrücke unendlichen Grades in $a(\varkappa')$, $a^+(\varkappa')$ sind, je nachdem, ob T von zweitem oder höherem Grad in $a(\varkappa)$, $a^+(\varkappa)$ ist.

RIASSUNTO (*)

Si esamina fin dove si possa costruire la teoria quantistica delle particelle elementari servendosi solo del seguente principio: « Esiste una rappresentazione irriducibile dell'algebra degli operatori di creazione e di distruzione a^+, a, e partendo da questi si possono costruire le trasformazioni infinitesime del gruppo inomogeneo di Lorenz. La rappresentazione del gruppo inomogeneo di Lorentz si può poi spezzare in rappresentazioni irriducibili classificate da WIGNER e descriventi differenti specie di particelle elementari. Per mezzo di una trasformazione unitaria degli a^+, a, si introduce la dipendenza di tutti gli operatori dallo spazio-tempo. I tensori energia-quantità di moto e i vettori carica-corrente si trovano senza far ricorso ai campi. Si possono anche impostare campi ed equazioni di campo corrispondenti alle varie specie di particelle, ma non sono indispensabili. Si introducono le interazioni come trasformazioni unitarie degli a^+, a, mescolando le rappresentazioni irriducibili del gruppo di Lorentz. Ciò corrisponde a termini generalmente non locali del formalismo di Lagrange.

(*) *Traduzione a cura della Redazione.*

Elastic Scattering and Intrinsic Structure of Elementary Particles.

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(ricevuto il 21 Marzo 1958)

Summary. — Experimental data (1) on elastic scattering of π^--mesons on protons with energy $E = 1.3$ GeV have been analysed. It is shown that from the analysis of their angular distribution it is possible to determine the root-mean-square radius and to get the data about the distribution of matter inside the nucleon. The root-mean-square « pion radius » is found to be equal to $\sqrt{\langle r^2 \rangle} = (0.82 \pm 0.06) \cdot 10^{-13}$ cm. In conclusion a possible experimental criterion of the existence of an elementary length is considered.

1. – Introduction.

Particle structure is determined by studying the elastic scattering of some rays by this particle. When observing such scattering in particle ensembles we get the « mean optical image » of the particle from which one may obtain the space-time picture of the intrinsic structure of the particle with respect to the selected rays. We should use shorter waves, if we intend to obtain more detailed information about particle structure.

In Sect. **4** we shall discuss the principle limits for the construction of such optical image of the particle and the limits of applicability of the space structure of elementary particles.

The experiments on the elastic scattering of fast electrons on a nucleon carried out by Hofstadter's group which allowed to determine the form-factor of

(1) M. Chretien, I. Leitner, N. P. Samios, M. Schwartz and J. Steinberger: *Phys. Rev.*, **108**, 383 (1957).

the electric charge and the nucleon's magnetic moment [2,3] are presently the only example of measurements of the structure of elementary particles. However, the study of the elastic scattering of fast particles of other kinds also gives the possibility for obtaining information about the nucleon's and nucleus' structure. This information contributes to the data obtained from the electron scattering experiments which, strictly speaking, in their turn give information only about the distribution of the electric charges and currents inside the particles under investigation, i.e., about the « electric » particle structure.

Further, a detailed analysis of 1.3 GeV π-meson scattering on proton is given [1]. This analysis, as it will be shown, makes it possible to get information both about the « nuclear » structure of a nucleon and about its « nuclear » or « pion » radius.

2. – Phase-shift analysis.

As is known, the differential cross-section of the elastically scattered particles may be presented in the form (*):

$$\frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\Omega} = \frac{\lambda\!\!\!^{-2}}{4}\left|\sum_{l=0}^{\infty}(2l+1)(1-\beta_l)\mathcal{P}_l(\cos\theta)\right|^2. \tag{1}$$

Here usual notations are used, in particualr, $\beta_l = \exp[+2i\eta_l]$, where η_l is the complex scattered wave phase.

On the basis of theoretical considerations [4] and from the direct comparison of calculation with experimental data [5-9] it follows, that in sufficiently high energy regions $E > E^*$ ($E^* \simeq 1$ GeV for π^--mesons and 5 GeV for nucleons) the real part of the phase may be put equal to zero with sufficient

(2) E. E. Chambers and R. Hofstadter: *CERN Symposium*, **7**, 295 (1956); R. Hofstadter: *Rev. Mod. Phys.*, **28**, 214 (1956).

(3) D. R. Yennie, M. M. Lévy and D. G. Ravenhall: *Rev. Mod. Phys.*, **29**, 144 (1957).

(*) For simplicity we do not take into account spin dependence of the interaction and neglect the « charge-exchange » process. (Compare (5)).

(4) S. Z. Belen'kij: *Žu. Ėksper. Teor. Fiz.*, **33**, 1248 (1957).

(5) D. Ito and S. Minami: *Progr. Theor. Phys.*, **14**, 198 (1955).

(6) C. S. Belen'kij: *Žu. Ėksper. Teor. Fiz.*, **30**, 983 (1956).

(7) D. Ito, T. Kobayashi, M. Yamazaki and S. Minami: *Progr. Theor. Phys.*, **18**, 264 (1957).

(8) V. G. Grišin and I. S. Saitov: *Žu. Ėksper. Teor. Fiz.*, **33**, 1051 (1957).

(9) V. G. Grišin, I. S. Saitov and I. V. Čuvilo: *Žu. Ėksper. Teor. Fiz.* (in print).

accuracy. In this case the quantity β_l will be real, due to this fact the phase shift analysis is considerably simplified (6).

The values of the function

$$I(l) = 2Im\eta_l = -\ln\beta_l \tag{2}$$

are given in Fig. 1 in the form of a hystogram.

To calculate the values between the extreme experimental values of the differential cross-section of the diffraction scattering $d\sigma_d(\theta)/d\Omega$, the curves with the largest and the smallest curvatures were plotted from (1) (cf. Fig. 3). The elastic scattering cross-section at zero angle in this case was normalized to the total cross-section $\sigma_t = \sigma_{el} + \sigma_{in} = (33.2 \pm 3)$ mb according to the optical theorem (10).

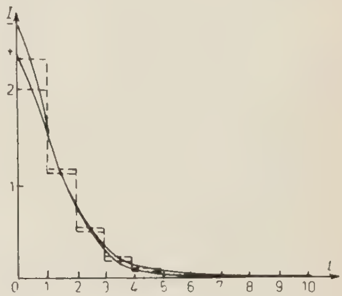

Fig. 1. – The histogram of the values of the function $I(l) = 2\,\mathrm{Im}\,\eta_l$ calculated for the extreme experimental values of the differential diffraction scattering cross-section. Solid curves are drawn through the centres of the histogram rectangles.

In accordance with so plotted curves two hystograms are given in Fig. 1. Solid curves are drawn through the rectangular centres of the hystograms.

The cross-section of the non-diffraction elastic scattering $d\sigma_{nd}/d\Omega = d\sigma_{el}/d\Omega - d\sigma_d/d\Omega$ at the energy $E = 1.3$ GeV in $(\pi^-\text{-p})$-collision is only some per cent of $d\sigma_d/d\Omega$ in the angle region $\theta \lesssim 40°$ and with good accuracy one may assume that the functions $I(l)$ in Fig. 1 concern only the pure diffraction scattering.

The error might appear due to the big angles region, where $d\sigma_{nd}/d\Omega \gg d\sigma_d/d\Omega$. However, the approximation $d\sigma_d(\theta)/d\Omega$ in this region of rapidly falling function excludes the isotropic non-diffraction scattering (*) (see also Sect. 3).

In accordance with the values $I(l)$ (see Fig. 1) the cross-section were calculated:

$$\sigma_t = \sigma_{in} + \sigma_d\,; \qquad \sigma_{in} = \sum_{l=0}^{10} \sigma_{in}(l)\,; \qquad \sigma_d = \sum_{l=0}^{10} \sigma_d(l)\,, \tag{3}$$

which are in good agreement with the experimental values:

$$\begin{aligned} \sigma_{in} &= (24.6 \div 29.0)\ \text{mb}\,; & \sigma_{in}^{exp} &= 26\ \text{mb}\,,\\ \sigma_d &= (\ 7.7 \div \ 7.9)\ \text{mb}\,; & \sigma_d^{exp} &= (7.5 \pm 1.2)\ \text{mb}\,,\\ \sigma_t &= (32.3 \div 36.9)\ \text{mb}\,; & \sigma_t^{exp} &= (33.2 \pm 3)\ \text{mb}\,. \end{aligned}$$

(*) The non-diffraction cross-section σ_{nd} is $\sim 15\%$ of σ_{in} due to great angles.

(10) L. I. LAPIDUS: *Žu. Ėksper. Teor. Fiz.*, **31**, 1099 (1956).

The calculated values

$$\Delta_{in}(l) = \sigma_{in}(l)/\sigma_t \qquad \text{and} \qquad \Delta_d(l) = \sigma_d(l)/\sigma_t \qquad \text{(in per cent)}$$

are given in Fig. 2.

It is seen from this figure that with $l > 6$-7 partial cross-sections rapidly decrease with the increase of l.

The angular distribution of elastically scattered particles reconstructed in consistence with (1) by the first ten values are in good agreement with the initial curves. The insignificant contribution of terms with great values of l, which were not taken into account, is due to the fact that pion nucleon innteractio is a short-range one.

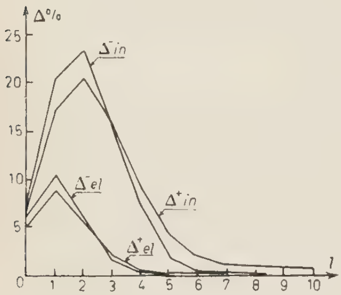

Fig. 2. – Relative contributions of the partial absorption cross-sections $\Delta_{in}(l) = \sigma_{in}(l)/\sigma_t$ and of the partial diffraction cross-sections $\Delta_d(l) = \sigma_d(l)/\sigma_t$ (in per cent). The indices « + » and « — » distinguish respectively the curves drawn for the cases of the differential diffraction scattering cross-section with the largest and the smallest curvatures. (See Fig. 3).

3. – Quasi-classical approximation and proton structure.

At high energies of the scattering particles, when the wave length $\lambda\!\!\!^{-}$ becomes considerably smaller in comparison with the dimensions of the scattering system and the relative change of the absorption coefficient in nuclear matter in the interval λ is $\Delta K/K \ll 1$, the quasi-classical approximation is applicable with good accuracy. In our case $\lambda\!\!\!^{-} = 0.28 \cdot 10^{-13}$ cm and some times less than the nucleon dimensions. Using the values $I(l)$ according to the well-known formulas [11] and assuming that the nucleon is purely absorbing and $r \simeq \lambda\!\!\!^{-}\sqrt{l\,(l+1)} \simeq \lambda\!\!\!^{-} l$, the cross-sections $\sigma_{in} = (25.5 \pm 1.5)$ mb; $\sigma_d = (7.4 \pm 0.1)$ mb were calculated. The angular distribution of elastically scattered particles is represented, in Fig. 3, by the dotted lines calculated in accordance with [11]. Solid curves designate the extreme values of experimental angular distribution from [1] with the largest and the smallest curvatures.

The good agreement of the calculated magnitudes with the corresponding ones calculated in the previous section and with their experimental values may be considered as one of the justifications of the further application of

(11) S. FERNBACH, R. SERBER and T. B. TAYLOR: *Phys. Rev.*, **75**, 1352 (1949).

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the quasi-classical approximation. Using this approximation, from the integral equation determining the imaginary part of the phase

$$I(\varrho) = \int\limits_0^{\sqrt{L^2-\varrho^2}} K(\sqrt{\varrho^2 + S^2})\,\mathrm{d}S\,; \qquad \varrho \simeq \lambda\!\!\!^{-}\sqrt{l(l+1)} \simeq \lambda\!\!\!^{-} l\,, \tag{4}$$

(here $L = l_{\max}$) we can calculate the pion absorption coefficient in nucleons as a function of the distance r from the nucleon centre using the known values of $I(l)$. For this purpose we rewrite equation (4) in the form:

$$I(\varrho) = \int K(r) Q(r, \varrho)\,\mathrm{d}r\,, \tag{5}$$

where

$$Q(r, \varrho) = \begin{cases} r/\sqrt{r^2 - \varrho^2} & \text{for} \quad r > \varrho\,, \\ 0 & \text{for} \quad r \leqslant \varrho\,. \end{cases}$$

For the numerical solution it is suitable to present (5) in the form:

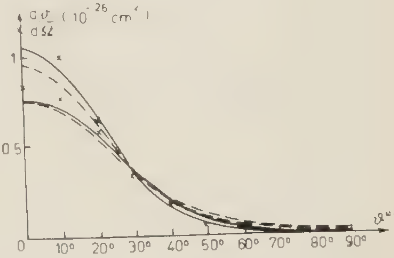

Fig. 3. – Solid curves show the extreme experimental values of the differential diffraction scattering cross-section (with the largest or the smallest curvatures). Dashed curves show the angular distribution of the diffraction scattering which was calculated according to the optical model formulae [11]. Two curves correspond to the two curves for $I(l)$ (see Fig. 1).

$$I_j = \sum_{i:j-1}^{n} k_i P_{ij}\,, \tag{7}$$

where

$$K_i = K(r_i)\,; \qquad P_{ij} = Q(r_i;\, \varrho_j)\frac{L}{n}\,; \qquad I_j = I[l(\varrho_j)]\,;$$

$\varrho_j = (j - \frac{1}{2})L/n$ being the mean point of the j-interval.

This linear equation system has « triangular form » in virtue of (6) and the solution may be easily found by successive substitutions. The function $K(r)$ thus calculated is represented in Fig. 4. This function determines the « pion structure » of a proton averaged over the space interval $\Delta r \sim \lambda\!\!\!^{-}$.

For the root-mean-square « pion radius » of a proton

$$\langle r^2 \rangle = \int\limits_0^L r^4 K(r)\,\mathrm{d}r \Big/ \int\limits_0^L r^2 K(r)\,\mathrm{d}r\,, \tag{8}$$

the following value was obtained

$$\sqrt{\langle r^2 \rangle} = (0.82 \pm 0.06)\cdot 10^{-13}\ \text{cm}\,, \tag{9}$$

which coincides with the « electromagnetic radius » of a proton obtained from Hofstadter's group experiments [2,3]. As is seen from Fig. 4, the absorption coefficient essentially increases in the central region of the proton. However, the values in this region are not quite precisely determined, as they are dependent on the approximation of $d\sigma_d(\theta)/d\Omega$ in the large angle interval.

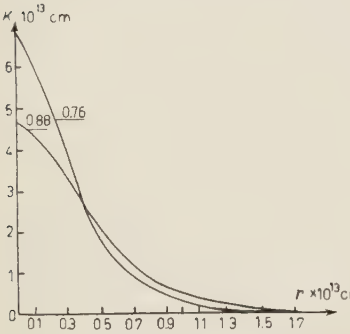

Fig. 4. – The absorption coefficient $K = K(r)$ as a function of the distance from the nucleon centre. Two curves correspond to the two extreme experimental values of the angular distribution of the diffraction scattering (with the largest or the smalles curvatures).

This inaccuracy in the separation of the diffraction scattering decreases rapidly with the energy increase, as the fraction of the non-diffraction elastic processes becomes negligible.

So with $E = 5$ GeV, $\sigma_{nd} = 0.06\,\%\,\sigma_{in}$; with $E = 7$ GeV, $\sigma_{nd} = 0.014\,\%\,\sigma_{in}$ (calculation according to the statistical theory [12]).

If in the peripheral regions of the proton π-mesons are mainly present one may assume that

$$K(r) = K\cdot\varrho(r)\,, \tag{10}$$

where K is the energy dependent coefficient of the meson absorption by the peripheral π-meson field, and $\varrho(r)$ is the mean density of the π-meson cloud near the point r. Within the accuracy of the experimental data the analytic form of $\varrho(r)$ may be approximated by different curves of the type described in [2].

In the central regions of the nucleon $K(r)$ is very likely to be determined by the other kinds of particles (nucleons, hyperons, K-mesons) and the formula (10) is not applicable.

4. – A possible experimental criterion for the existence of an elementary length.

During the last years the idea that there may really exist a limit of applicability of the conventional space-time description of the particle structure connected with the existence of a certain « elementary length » has been fre-

[12] V. S. Barašenkov and V. M. Maltsev: *Acta Phys. Pol.* (in print).

quently suggested. This idea was expressed in different versions of the theory of « non-local fields » or « non-local interaction » (see, e.g., [13]). Such theoretical schemes lead to form-factors which weaken the interaction for the short waves. In this way one might hope to eliminate the divergencies from the modern quantum theory due to very short wavelengths.

However, this point may be the weakest one in the « non-local » theories (*). G. V. WATAGHIN and E. FERMI were the first to notice in their statistical theory of multiple production [15] that at high energies the interaction becomes not a weak, but, on the contrary, a strong one. The calculations show that the weak (in the sense of the generally accepted classification) interaction of Fermi type (ν, e, μ) also becomes strong at high energies (cross-section $> \lambda\!\!\!^{-2}$) [16-18].

Now we should like to put a question: under what conditions from a purely empirical standpoint would it be possible to speak about the non-locality? Evidently, these conditions should occur when it would become impossible to use the elastic scattering of particles as a means of studying their structure. Thus, the matter depends upon the asymptotic behaviour of the cross-sections at high energies.

If with $\lambda \to 0$ all the elastic scatterings for a certain internal region R will tend to the diffraction scattering on a « black sphere » of radius R, the elastic scattering will cease to give information about the intrinsic structure of this region and the maximum information will be limited by the data about the outer dimensions of the « sphere ».

The scattering cross-section which is due to the process in this region will be equal to πR^2 ($\gg \pi\lambda\!\!\!^{-2}$) and equals the corresponding inelastic scattering.

In this case, instead of the description of the space-time structure, the problem about possible ways of particle transformation will become important.

The magnitude of R from this point of view is the same length scale which determines the real non-locality, i.e., the limit of the applicability of space-time description of the particle structure.

It can be seen from the analysis of the pion scattering on protons that the central region of the nucleon tends to appear as « black ».

(13) D. I. BLOHINCEV: *Usp. Fiz. Nauk*, **61**, 137 (1957); V. S. BARAŠENKOV: *Nuovo Cimento*, **5**, 1469 (1957).

(*) This circumstance was also noted by M. A. MARKOV [14].

(14) M. A. MARKOV: *Usp. Fiz. Nauk*, **51**, 317 (1953).

(15) G. WATAGHIN: *Symposium sobre raios cosmicos*, Rio de Janeiro (August 4-8, 1941); E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1949).

(16) D. I. BLOHINCEV: *Usp. Fiz. Nauk*, **62**, 381 (1957).

(17) I. E. TAMM: *Žu. Èksper. Teor. Fiz.*, **32**, 178 (1957).

(18) V. S. BARAŠENKOV: *Proceedings of the Conference in Padua-Venice* (1957); *Nucl. Phys.* (in print).

From the point of view given here the further study of the energetic dependence of elastic pion scattering may be of principal importance.

It can be said that the dimension of the non-locality radius R must not be a universal length S_0; it may depend upon the kind of the interaction. The minimum scale of the space-time description R determined by the dimensions of the « black sphere » may be introduced into the theory in a relativistic in variant way. Indeed, the scattered wave phase η_e is an invariant. We intend to consider it as a function of two invariants ([19]):

$$(11) \qquad D = \frac{\Gamma_\mu \Gamma^\mu}{\mathcal{P}_\mu \mathcal{P}^\mu} \qquad \text{and} \qquad F = P_\mu P^\mu + \frac{(P_\mu \mathcal{P}^\mu)^2}{\mathcal{P}^\mu \mathcal{P}_\mu}.$$

Here $\mathcal{P}_\mu$ is the four-dimensional energy-momentum vector of the whole system whereas P_μ is the same for the relative motion of the incident particle and the particle of the scatterer, and finally

$$(12) \qquad \Gamma_\mu = \varepsilon_{\mu\nu\alpha\beta} M_{\alpha\beta} \mathcal{P}_\nu .$$

Here $\varepsilon_{\mu\nu\alpha\beta}$ is the fully antisymmetrical unit tensor of fourth rank, $M_{\beta\alpha}$ is the momentum antisymmetrical tensor. Using these invariants the « black sphere » may be determined as follows:

$$(13) \qquad \eta_e = 0 \quad \text{if} \quad D/F > R^2 \qquad \text{and} \qquad \eta_e = +i\infty, \quad \text{if} \quad D/F < R^2,$$

for $F \geqslant P_0^2$, where P_0 is the great value of the momentum at which the opacity occurs.

The quantity D/F is an operator, therefore, the inequalities (13) are determined for its eigenvalues.

It can be easily seen that in the center of mass system ($\mathcal{P} = 0$)

$$D/F = \frac{\boldsymbol{M}^2}{\boldsymbol{P}_0^2} = \frac{\hbar^2 l(l+1)}{\boldsymbol{P}_0^2},$$

where $\boldsymbol{M}$ is the three-dimensional angular momentum, while $\boldsymbol{P}_0$ is the three-dimensional momentum of the relative motion. D/F determines the collision parameter in a relativistic-invariant way.

Note in conclusion that in perturbation theory there are known « propagation functions » which lead to divergencies in the region of great frequencies. These functions are constructed with the help of the plane waves which were used as a zero approximation. Meanwhile at great frequencies

([19]) Yu M. Širokov: *Žu. Èksper. Teor. Fiz.*, **21**, 748 (1951); M. A. Markov: *Dokl. Akad. Nauk SSSR*, **101**, 449 (1955).

of the field in the presence of particles the plane wave will be quite a bad approximation due, to the diffraction scattering. Instead of the plane wave one ought to take a series expansion accounting for the sharp change of the wave field at great relative momenta of the interacting particles. This gives rise to the relativistic invariant cut-off form-factors. However, these form-factors are not due to weakening of the interaction at high frequencies, as it is assumed in the conventional non local theories but, on the contrary, to its strengthening. As a whole, phenomenologically, this factor takes into account the intensive inelastic processes of any origin which occur at high energies.

RIASSUNTO (*)

Si sono analizzati dati sperimentali sullo scattering elastico di mesoni π su protoni di $E = 1.3$ GeV. Si dimostra che dall'analisi della loro distribuzione angolare è possibile determinare il raggio quadratico medio ed ottenere dati sulla distribuzione della materia all'interno del nucleone. Il raggio quadratico medio del pione risulta uguale a $\sqrt{\langle r^2 \rangle} = (0.82 \pm 0.06) \cdot 10^{-13}$ cm. Nelle conclusioni si considera un possibile criterio sperimentale per la verifica dell'esistenza di una lunghezza elementare.

(*) *Traduzione a cura della Redazione.*

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Correlated Polarization of Muons in $K_{\mu3}$-Decay (*).

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(ricevuto il 25 Marzo 1958)

Summary. — The relativistic density matrix formalism developed by Stapp is used to calculate the polarization vector of muons from $K_{\mu3}$-decay and the depolarizing effects due to high energy atomic collisions.

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1. – The polarization vectors of muons produced in $K_{\mu3}$-decay is derived in a straightforward way using the relativistic density matrix formalism developed by Stapp (1). The relativistic density matrix in momentum space may be written in the form (×):

$$\varrho(p) = \frac{1 \mp \gamma_5 a}{2} \Lambda_\pm \gamma_0 \lambda(p) , \tag{1}$$

where $\Lambda_\pm = (p \pm m)/2m$ is the projection operator for particle and antiparticle states respectively and a_μ is a four-vector such that $p \cdot a = 0$ and $0 \leqslant -a^2 \leqslant 1$. The probability of finding particles with momentum within the interval $(\boldsymbol{p}, \boldsymbol{p} - \mathrm{d}\boldsymbol{p})$ is $(m/E)\lambda(p)\,\mathrm{d}^3p$. The expectation value of an operator $O(p)$ is given by:

$$\langle O \rangle = \int \mathrm{Tr}\,(\varrho O)\,\frac{m}{E}\,\mathrm{d}^3p\,. \tag{2}$$

(*) This work has been done under the auspices of the Conselho Nacional de Pesquisas of Brazil.

(+) On leave of absence from the Centro Brasileiro de Pesquisas Fisicas.

(1) H. P. Stapp: *Phys. Rev.*, **103**, 425 (1956).

(×) This matrix is actually only the diagonal part with respect to the momentum variables of the density matrix in momentum representation.

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The « polarization vector » for particles of given momentum is the expectation value of the spin-matrix $\boldsymbol{\sigma}$ in the centre of mass system. On can see that:

$$\text{(3)} \qquad \boldsymbol{P} = \boldsymbol{a} - a_0 \boldsymbol{p}/(E+m)\,,$$

For the decay process $K^{\mp}_{\mu3} \to \mu^{\mp} + \nu + \pi^0$, assuming a primary Fermi interaction (²), one has:

$$\text{(4)} \qquad \varrho(p)\frac{m}{E}\,\mathrm{d}^3p = \Lambda_{\pm}\bar{Q}p_\nu Q \Lambda_{\pm}\gamma_0(2\pi)^4\,\delta(p_f - p_i)\,\mathrm{d}\tau\,,$$

where p_f and p_i are the total momenta in the initial and final states respectively, $\mathrm{d}\tau$ is an element of phase space volume and $Q = \frac{1}{2}(1-\gamma_5)Q(g) + \frac{1}{2}(1+\gamma_5)Q(f)$, where

$$\text{(5)} \qquad Q(g) = \frac{1}{M}\,g_s - \frac{1}{M^2}\left(g_V\,p_K + g_{V'}\,p_\pi\right) + \frac{i}{M^3}\,g_T\sigma_{\mu\nu}p_K^\mu p_\pi^\nu\,.$$

Re-writing (4) so as to bring it into the form given by (1), one finds:

$$\text{(6)} \qquad \boldsymbol{P} = (\boldsymbol{A}_g - \boldsymbol{A}_f)/(\Delta_g + \Delta_f)\,,$$

where

$$\text{(7)} \qquad \boldsymbol{A} = -\Big\{[|X_s|^2 p\cdot p_\nu - |X_V|^2(2EE_\nu - p\cdot p_\nu)]\boldsymbol{p}/E - [(|X_s|^2 + |X_V|^2)m +$$

$$+ 2\,\mathrm{Re}\,(X_s X_V^*)E]\left(\boldsymbol{p}_{\nu N} + \frac{m}{E}\,\boldsymbol{p}_{\nu L}\right) - 2\,\mathrm{Im}\,(X_s X_V^*)\boldsymbol{p}\times\boldsymbol{p}_\nu\Big\}\,,$$

and

$$\text{(8)} \qquad \Delta = |X_s|^2 p\cdot p_\nu + |X_V|^2(2EE_\nu - p\cdot p_\nu) + 2\,\mathrm{Re}\,(X_s X_V^*)mE_\nu\,;$$

where $\boldsymbol{p}_{\nu N}$ and $\boldsymbol{p}_{\nu L}$ stand for components $\boldsymbol{p}_\nu$ normal and parallel to $\boldsymbol{p}$, and

$$\text{(9)} \qquad X_s = \frac{1}{M}\left(g_s - \frac{m}{M}\,g_{V'} + \frac{E-E_\nu}{M}\,g_T\right); \qquad X_\nu = \frac{1}{M}\left(g_V + g_{V'} - \frac{m}{M}\,g_T\right);$$

with entirely analogous expressions with the g's replaced by the f's. Analogous expressions have been obtained by CHARAP (³), using spin projection operators (⁴).

(²) S. FURUICHI, T. KODAMA, S. OGAWA, Y. SUGAWARA, A. WAKASA and M. YONEZAWA: *Progr. Theor. Phys. (Japan)*, **17**, 89 (1957).

(³) J. M. CHARAP: *Some aspects of* $K_{\mu3}$*-decay*, to be published.

(⁴) L. MICHEL and A. S. WIGHTMAN: *Phys. Rev.*, **98**, 1190 (1955); H. A. TOLHOEK: *Rev. Mod. Phys.*, **28**, 277 (1956).

The two-component theory (5) corresponds to setting either all g's or all f's equal to zero. In this case for a given final configuration of the momenta the polarization is complete ($P^2 = 1$). Therefore the analysis of correlated measurements in the decay chain $K_{\mu 3}$-μ-e is a very promising approach to the investigation of the decay mechanism. On the basis of the law of conservation of leptons, the alternative $g = 0$ should be excluded.

As shown in a previous paper (6), all parameters must be real for invariance of the interactions under time reversal. Direct evidence of violation of this principle would be provided by observation of a polarization normal to the plane of decay. However, this results from interference between S, V, T (or P, A, T) couplings and could not occur if the Fermi interaction were a $(V+A)$ combination. Correlated polarization provides an independent way of determining the best set of parameters g's and f's. In the analysis of the pion energy dependence of these parameters it supplies information additional to that from angular correlation experiments at fixed pion energies.

2. – The expressions for normal and longitudinal polarization must be corrected by factors which take into account the depolarization. This may easily be done in the case when the depolarizing factors are identical and independent of the primary muon energy. The main cause of depolarization, which could depend on the spin orientation and the muon energy is atomic collisions at high energies.

Let M be the matrix element for collision, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ the initial and final momenta, ϱ_1 and ϱ_2 the density matrix in the initial and final states. One can write:

$$\varrho_2\gamma_0 = \Lambda_{2+}\overline{M}\varrho_1\gamma_0 M\Lambda_{2+}\,. \tag{10}$$

Transforming the density matrices to the rest system of the particle, the above relation becomes:

$$\varrho_2^0 = \frac{\gamma_0+1}{2}\, L_2\overline{M}\,\overline{L}_1\varrho_1^0 L_1 M \overline{L}_2\, \frac{\gamma_0+1}{2}\,, \tag{11}$$

where $L = (\gamma_0 p + m)/[2m(E+m)]^{\frac{1}{2}}$ is a Lorentz translation of velocity $-\boldsymbol{p}/E$. Since one can write:

$$\frac{\gamma_0+1}{2}\, L_1 M L_2\, \frac{\gamma_0+1}{2} = (\alpha + \boldsymbol{\sigma}\cdot\boldsymbol{\beta})\, \frac{\gamma_0+1}{2}\,, \tag{12}$$

(5) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957); A. SALAM: *Nuovo Cimento*, **5**, 299 (1957); L. LANDAU: *Nucl. Phys.*, **3**, 127 (1957).

(6) S. W. MAC DOWELL: *Nuovo Cimento*, **6**, 1445 (1957).

it follows that:

$$(13)\qquad \varrho_2^0 = (\alpha^* + \boldsymbol{\sigma}\cdot\boldsymbol{\beta}^*)\,\frac{1+\boldsymbol{\sigma}\cdot\boldsymbol{P}_1}{2}\,(\alpha + \boldsymbol{\sigma}\cdot\boldsymbol{\beta})\,\gamma_0\,\frac{-1}{2} =$$

$$= \frac{1}{2}\left\{(\alpha\alpha^* + \boldsymbol{\beta}\boldsymbol{\beta}^*)\cdot(1+\boldsymbol{\sigma}\cdot\boldsymbol{P}_1) + \boldsymbol{P}_1\cdot[-i\boldsymbol{\beta}\times\boldsymbol{\beta}^* + 2\,\mathrm{Re}\,\alpha\boldsymbol{\beta}^*] + \right.$$

$$\left. + \boldsymbol{\sigma}\cdot[-i\boldsymbol{\beta}\times\boldsymbol{\beta}^* + 2\,\mathrm{Re}\,(\alpha\boldsymbol{\beta}^* + i\alpha\boldsymbol{\beta}^*\times\boldsymbol{P}_1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}^*\boldsymbol{P}_1 + \boldsymbol{\beta}\cdot\boldsymbol{P}_1\boldsymbol{\beta}^*)]\right\}\frac{\gamma_0+1}{2}\,.$$

Supposing that the particles are slowing down in a medium with n scattering centres per unit volume, the average depolarization per unit path is the product of n (cross-section) by the average depolarization per collision. All the terms in (13) of the form $\alpha\boldsymbol{\beta}^*$ and $\boldsymbol{\beta}\times\boldsymbol{\beta}^*$ vanish on the average; one then obtains:

$$(14)\qquad -\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}x} = 2n\,\frac{m}{p}\int[(\boldsymbol{\beta}\times\boldsymbol{P})\times\boldsymbol{\beta}^*\,\mathrm{d}\tau]_{\mathrm{Av}}\,,$$

where the result of integration over phase space is averaged over the incident directions. The normal and longitudinal depolarization are then given by

$$(15,a)\qquad -\frac{1}{P_{\mathrm{N}}}\frac{\mathrm{d}P_{\mathrm{N}}}{\mathrm{d}x} = n\,\frac{m}{p}\int[\boldsymbol{\beta}\cdot\boldsymbol{\beta}^* + |\boldsymbol{\beta}\cdot\boldsymbol{\epsilon}_1|^2 + \frac{1}{2}(\boldsymbol{\beta}\cdot\boldsymbol{\beta}^* - 3|\boldsymbol{\beta}\cdot\boldsymbol{\epsilon}_1|^2)\langle\sin^2\Theta\rangle]\,\mathrm{d}\tau\,,$$

$$(15,b)\qquad -\frac{1}{P_{\mathrm{L}}}\frac{\mathrm{d}P_{\mathrm{L}}}{\mathrm{d}x} = 2n\,\frac{m}{p}\int[\boldsymbol{\beta}\cdot\boldsymbol{\beta}^* - |\boldsymbol{\beta}\cdot\boldsymbol{\epsilon}_1|^2 - \frac{1}{2}(\boldsymbol{\beta}\cdot\boldsymbol{\beta}^* - 3|\boldsymbol{\beta}\cdot\boldsymbol{\epsilon}_1|^2)\langle\sin^2\Theta\rangle]\,\mathrm{d}\tau\,,$$

where $\boldsymbol{\epsilon}_1 = \boldsymbol{p}_1/p_1$ and Θ is the deviation of $\boldsymbol{\epsilon}_1$, from the initial direction after the particle has travelled a distance x. At high energies the mean square deviation is small and the term in Θ is negligible.

For elastic scattering neglecting the nuclear recoil and using first Born approximation, one obtains

$$(16)\qquad -\frac{2}{P_{\mathrm{N}}}\frac{\mathrm{d}P_{\mathrm{N}}}{\mathrm{d}x} = -\frac{1}{P_{\mathrm{L}}}\frac{\mathrm{d}P_{\mathrm{L}}}{\mathrm{d}x} = 4\pi n\,\frac{Z^2e^4}{(E+m)^2}\ln\left(\frac{\sin\theta_{\mathrm{max}}/2}{\sin\theta_{\mathrm{min}}/2}\right),$$

where θ_{min}, estimated by means of the Thomas-Fermi statistical model, is given by $\sin(\theta_{\mathrm{min}}/2) = Z^{\frac{1}{3}}m_e e^2/p$ and $\sin(\theta_{\mathrm{max}}/2) \leqslant 1/r_n p$, where $r_n = r_0 A^{\frac{1}{3}}$ is the nuclear radius. The value of θ_{max} may also be fixed by the experimental conditions. For inelastic scattering one can suppose the muon to be interacting directly with one electron, which may be considered as free and initially at rest. The results in first order perturbation theory are:

$$(17,a)\qquad -\frac{1}{P_{\mathrm{N}}}\frac{\mathrm{d}P_{\mathrm{N}}}{\mathrm{d}x} = \pi n\,\frac{Ze^4}{(E+m)^2}\left\{\ln\left(\frac{2m_e p^2}{ZIm^2}\right) + \frac{1}{m^2}[p^2 + mE + (m+E)^2]\right\}.$$

$$\frac{1}{P_{\rm L}}\frac{\mathrm{d}P_{\rm L}}{\mathrm{d}x} = 2\pi\frac{Ze^4}{(E+m)^2}\left\{\ln\left(\frac{2m_e p^2}{ZIm^2}\right) + \frac{1}{m^2}\left[p^2 + mE\right]\right\}. \tag{17,b}$$

Dividing these results by the energy loss and integrating one obtains the depolarizing factor for these processes. For muons with initial kinetic energy ~ 130 MeV, slowed down to ~ 0.5 MeV in carbon, the depolarization due to atomic collisions is $\sim 1\%$. Appreciable depolarization occurs only at very low energies due to capture processes. Therefore one can expect that the depolarization will be almost isotropic and independent of the primary muon energy.

* * *

I wish to thank Professor PEIERLS for helpful discussions and encouragement.

RIASSUNTO (*)

Si fa uso del formalismo relativistico della matrice di densità per calcolare il vettore di polarizzazione dei muoni derivanti dal decadimento $K_{\mu 3}$ e gli effetti di depolarizzazione dovuti a collisioni atomiche di alta energia.

(*) *Traduzione a cura della Redazione.*

Quadratic Lagrangians and General Relativity.

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(ricevuto il 10 Aprile 1958)

Summary. — Quadratic invariants of the Riemann-Christoffel curvature tensor and its contractions in a four-dimensional Riemann space are used as the Lagrangians in three variational principles. The field equations are derived by treating the metric tensor and the arbitrary symmetric affine connection as independent variables (following the method of Palatini), and specializing to the Christoffel connection after the variation. It is shown that the field equations derived from two of these variational principles in this way have as a class of solutions all solutions of Einstein's equations with cosmological term, whilst all three sets of field equations are satisfied by the Schwarzschild metric and have vanishing divergence. This suggests alternative forms of the field equations for gravitation, quadratic in the Riemann-Christoffel tensor and with zero trace, which give the same results for the three «crucial tests» of general relativity as Einstein's equations $R_{ik}=0$.

1. – Introduction.

Quadratic Lagrangians formed from the Riemann-Christoffel curvature tensor and its contractions in a Riemannian space-time of four dimensions were first discussed by Weyl (1), Pauli (2) and Eddington (3), and later by Lanczos (4,5) in an attempt to include the electromagnetic field in Riemannian

(1) H. Weyl: *Sitz. d. Preuss. Akad. d. Wiss.*, p. 465 (1918); *Ann. der Phys.*, **59**, 101 (1919); *Phys. Zeits.*, **22**, 473 (1921).

(2) W. Pauli: *Phys. Zeits.*, **20**, 457 (1919).

(3) A. S. Eddington: *The Mathematical Theory of Relativity* (Cambridge, 1952), p. 141.

(4) C. Lanczos: *Ann. Math.*, **39**, 842 (1938).

(5) C. Lanczos: *Rev. Mod. Phys.*, **21**, 497 (1949); **29**, 337 (1957).

geometry. Further work has been carried out by GREGORY [6], BUCHDAHL [7] and ARNOWITT [8].

Three particular quadratic invariants, namely, R^2, $R_{ik}R^{ik}$ and $R_i{}^{\underset{\smile}{kmn}}R^i{}_{\underset{\smile}{kmn}}$ have been used as the Lagrangians in the integrals

$$I_1=\int R^2\sqrt{-g}\,\mathrm{d}\tau\,,\quad I_2=\int R_{ik}R^{ik}\sqrt{-g}\,\mathrm{d}\tau\,,\quad I_3=\int R_i{}^{\underset{\smile}{kmn}}R^i{}_{\underset{\smile}{kmn}}\sqrt{-g}\,\mathrm{d}\tau\,,$$

where g is the determinant of the metric tensor g_{ik} (assumed here to be symmetric), and the integration is carried out over the four-dimensional Riemannian space-time. A « bar » or « hook » under tensor indices means symmetry or skew-symmetry with respect to those indices. Provided the affine connection is assumed to be the Christoffel connection

$$\Gamma^s_{ik}=\begin{Bmatrix} s \\ ik \end{Bmatrix}=\frac{1}{2}\,g^{sn}(g_{ni,k}+g_{nk,i}-g_{ik,n})\,, \tag{1}$$

prior to the variation, the variation of these integrals subject to $\delta I_1=0$, $\delta I_2=0$ and $\delta I_3=0$, and carried out with respect to the variables g_{ik}, leads in each case to ten equations of the fourth-order for the ten g_{ik}. It has been shown, furthermore, that these three variations are not all independent (see LANCZOS [4]) and that

$$\delta(I_1-4I_2+I_3)=0$$

is an identity in a four-dimensional Riemannian space-time. The possibility of fourth-order equations being important in the « geometrodynamical » theory of gravitation and electromagnetism has recently been suggested by the work of MISNER and WHEELER [9].

Before considering an alternative approach to the derivation of the field equations from quadratic Lagrangians, we first indicate two different methods of deriving the field equations $R_{ik}=0$ of general relativity from a linear Lagrangian.

In the first place we may assume that the connection is the Christoffel connection given by (1) and consider the variation

$$\delta\int R\sqrt{-g}\,\mathrm{d}\tau=0\,. \tag{2}$$

[6] C. GREGORY: *Phys. Rev.*, **72**, 72 (1947).

[7] H. BUCHDAHL: *Quart. Journ. Math.* (*Oxford*), **19**, 150 (1948); *Journ. Lond. Math. Soc.*, **26**, 139, 150 (1951); *Proc. Nat. Acad. Sci.*, **34**, 66 (1948).

[8] R. L. ARNOWITT: *Phys. Rev.*, **105**, 735 (1957).

[9] C. W. MISNER and J. A. WHEELER: *Ann. of Phys.*, **2**, 525 (1957).

This leads to the equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = 0 \,, \tag{3}$$

which imply

$$R_{ik} = 0 \,. \tag{4}$$

Since R_{ik} is formed from the Christoffel connection it is symmetric, giving ten equations for the ten g_{ik}.

On the other hand following the method of Palatini, we need make no *a priori* assumption about the relation between the g_{ik} and the affine connection. Treating the g_{ik} and the arbitrary symmetric connection Γ^s_{ik} as independent variables in (2), we now find

$$R_{ik} = 0 \tag{5}$$

and

$$g_{mn;k} = 0 \,. \tag{6}$$

Equation (6) shows the affine connection to be the Christoffel connection, and substituting this in (5) leads to the same ten field equations for the g_{ik} as (4). In the case of the variation $\delta\int R\sqrt{-g}\,\mathrm{d}\tau = 0$ then, the two methods lead to the same equations for the g_{ik}. However, as we shall show in this paper, the variations

$$\delta I_1 = 0 \,, \qquad \delta I_2 = 0 \,, \qquad \delta I_3 = 0 \tag{7}$$

lead to different field equations for the g_{ik} depending on whether the connection is assumed to be the Christoffel connection before the variation or after.

In Sect. **2** the variation of the integrals is carried out, and in Sect. **3** the field equations corresponding to the special case of the Christoffel connection are discussed. Solutions of these field equations are given in Sect. **4**, and it is shown that the field equations from all three variational principles possess the Schwarzschild metric as a solution. This result suggests alternative forms for the field equations for the gravitational field in empty space which will give the same results for the three «crucial tests» of general relativity as Einstein's equations $R_{ik} = 0$, but which differ in mathematical form. Some of the implications of this result are discussed in Sect. **5**.

2. – Variation of the integrals.

We now assume that the Riemann-Christoffel curvature tensor is formed from an arbitrary symmetric connection Γ^s_{ik}, and perform the variation with respect to the g_{ik} and the Γ^s_{ik} independently. The following equations are

obtained:

Set (*A*).
$$\delta\int R^2\sqrt{-g}\,\mathrm{d}\tau = 0\;;$$

(8)
$$R(R_{\underline{ik}} - \tfrac{1}{4}g_{ik}R) = 0\,,$$

(9)
$$(Rg^{mn}\sqrt{-g})_{;s} = 0\,.$$

Set (*B*).
$$\delta\int R_{ik}R^{ik}\sqrt{-g}\,\mathrm{d}\tau = 0\;;$$

(10)
$$R_i^{\ s}R_{ks} + R_{sk}R^s_{\ i} - \tfrac{1}{2}g_{ik}R_{sm}R^{sm} = 0,$$

(11)
$$(R^{\underline{mn}}\sqrt{-g})_{;s} = 0\,.$$

Set (*C*).
$$\delta\int R_i^{\ k\underset{\smile}{mn}}R^i_{\ k\underset{\smile}{mn}}\sqrt{-g}\,\mathrm{d}\tau = 0\;;$$

(12)
$$-R^{is\underset{\smile}{mn}}R_{ks\underset{\smile}{mn}} + R^{si\underset{\smile}{mn}}R_{sk\underset{\smile}{mn}} + 2R^{sm\underset{\smile}{in}}R_{sm\underset{\smile}{kn}} - \tfrac{1}{2}\delta^i_kR_{sp\underset{\smile}{mn}}R^{sp\underset{\smile}{mn}} = 0\,,$$

(13)
$$(R_i^{\ \underline{m}\underset{\smile}{ns}}\sqrt{-g})_{;s} = 0\,.$$

These equations are derived with the help of the relation

(14)
$$\delta R^i_{\ .s\underset{\smile}{mn}} = -(\delta\Gamma^i_{sm})_{;n} + (\delta\Gamma^i_{sn})_{;m}\,.$$

The symmetries indicated in (11) and (13) arise since these equations are obtained by placing the coefficients of $\delta\Gamma^s_{mn}$ equal to zero in the variational principle.

3. – Specialization of the connection.

Although (9), (11) and (13) are each a set of forty equations for the forty components of the symmetric affine connection in terms of the g_{ik}, we shall now consider the consequences of choosing Γ^s_{ik} to be the Christoffel connection (as given by (1)). The field equations now become $\left(\text{where ; means covariant differentiation with respect to } \begin{Bmatrix} s \\ ik \end{Bmatrix}\right)$:

Set (*A*).

(15)
$$R^i_{\ k} - \tfrac{1}{4}\delta^i_kR = 0\,,$$

(16)
$$R_{,k} = 0\;;$$

or

$$R = 0 \,. \tag{17}$$

In virtue of the identity $(R^s{}_k - \frac{1}{2}\delta^s_k R)_{;s} = 0$, equations (16) show the vanishing divergence of (15). The alternative field equation (17) was originally proposed by LITTLEWOOD [10], but has been shown by PIRANI [11] to give an incorrect value for the advance of the perihelion of Mercury.

Set (B).

$$R^{is}R_{ks} - \tfrac{1}{4}\delta^i_k R_{sm}R^{sm} = 0 \,, \tag{18}$$

$$R^{ik}{}_{;s} = 0 \,. \tag{19}$$

Equations (18) are Rainich's equations (see MISNER and WHEELER [9]), and have vanishing divergence in virtue of (19).

Set (C).

$$R^{ismn}R_{ksmn} - \tfrac{1}{4}\delta^i_k R_{spmn}R^{spmn} = 0 \,, \tag{20}$$

$$R_i{}^{\underline{mns}}{}_{;s} = 0 \,. \tag{21}$$

Using the Bianchi identities, a sufficient condition that (20) has vanishing divergence is found to be

$$R_i{}^{mns}{}_{;s} = 0 \,. \tag{22}$$

Equation (21), which come from the variational principle together with the specialization of the connection, does not therefore give the vanishing divergence of (20) (unlike Set (A) and Set (B)), but is clearly satisfied when (22) is satisfied. Equation (22) may be written, again using the Bianchi identities, as

$$R_{ni;m} - R_{nm;i} = 0 \,, \tag{23}$$

and provided this equation is satisfied the first set of equations in (C) has vanishing divergence, whilst the second set is satisfied identically.

(10) D. E. LITTLEWOOD: *Proc. Camb. Phil. Soc.*, **49**, 90 (1953).
(11) F. A. E. PIRANI: *Proc. Camb. Phil. Soc.*, **51**, 535 (1955).

4. – Solution of field equations.

Certain solutions of the field equations discussed in Sect. 3 have been found.

Solution 1. – Equations (A), (B) and (C) are all satisfied by

$$R_{ikmn} = \lambda(g_{im}g_{kn} - g_{in}g_{km}) , \tag{24}$$

where λ is a constant. The form of the Riemann-Christoffel curvature tensor given by (24) expresses the fact that the Riemannian space is one of constant curvature.

Solution 2. – Equations (A) and (B) are satisfied by

$$R_{ik} - \tfrac{1}{2}g_{ik}(R - 2\lambda) = 0 , \tag{25}$$

where λ is a constant. Equations (25) are the field equations of general relativity with cosmological term and clearly contain the special case $R_{ik} = 0$. Hence all solutions of Einstein's equations satisfy (A) and (B).

Solution 3. – The first set of equations in (C), namely (20), are satisfied by the Schwarzschild metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) ,$$

as may be seen from the fact that the only non-vanishing components of the Riemann-Christoffel tensor for this metric are

$$R_{1212} , \quad R_{1414} , \quad R_{3131} , \quad R_{2424} , \quad R_{2323} , \quad R_{3434} .$$

It follows that

$$R^{ismn}R_{ksmn} = \delta^i_k f(m, r) , \tag{26}$$

where f is a function of m and r. Equations (20) also have vanishing divergence for this solution, since (23) is satisfied in virtue of the Schwarzschild metric being a solution of $R_{ik} = 0$.

5. – Conclusion.

We have shown that the field equations considered in Sect. 3 all possess the Schwarzschild metric as a solution and have vanishing divergences (at least for this solution, in the case of (C)). These equations will therefore

give the same results for the three « crucial tests » of general relativity (the bending of light, the advance of the perihelion of Mercury and the red-shift) as Einstein's equations

$$R_{ik} = 0 \,. \tag{27}$$

The differ, however, in mathematical form and complexity, and in particular sets (B) and (C) are second order, second degree equations in the g_{ik}, whereas Einstein's equations are second order, first degree. It would be of interest to study in more detail the differences between (27) and the field equations discussed here. For example, from the point of view of gravitational radiation theory, we might ask whether Birkhoff's theorem is true for the new field equations. Also what solutions are admitted when sources are present in the form of a given energy-momentum tensor?

It is hoped to discuss some of these problems in a later paper.

* * *

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RIASSUNTO (*)

Come lagrangiani, in tre principi variazionali si usano gli invarianti quadratici del tensore di curvatura di Riemann-Christoffel e le sue contrazioni in uno spazio riemanniamo quadridimensionale. Si ottengono le equazioni del campo trattando il tensore metrico e la connessione affine arbitraria come variabili indipendenti (secondo il metodo di Palatini) passando, dopo la variazione, alla connessione di Christoffel. Si dimostra che le equazioni del campo derivate in tal modo da due di questi principi variazionali hanno come classe di soluzioni tutte le soluzioni delle equazioni di Einstein contenenti un termine cosmologico, mentre i tre sistemi di equazioni del campo sono soddisfatti dalla metrica di Schwarzschild e hanno divergenza evanescente. Ciò suggerisce forme alternative delle equazioni del campo gravitazionale, quadratiche nel tensore di Riemann-Christoffel e con traccia zero, che danno per le tre « prove cruciali » della relatività generale gli stessi risultati delle equazioni di Einstein $R_{ik} = 0$.

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Polarization of Particles with Spin Unity.

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Summary. — The 3×3 matrix which describes the polarization of a beam of particles with spin unity is discussed. The degree of polarization is defined and the type of polarization discussed in terms of the vector and tensor that occur in the matrix. It is shown that the vector tends to coincide with a principal axis of the tensor. The equivalence of the appropriate special case with the Stokes parameter treatment of the polarization of electromagnetic radiation is indicated.

1. – Introduction.

It is well-known that the polarization of a beam of particles with two spin states (including the photon) can be described in terms of a density or polarization matrix having the form (1). The corresponding problem for particles with spin unity (where the photon can again be included) has received less attention [1].

In this paper some of aspects of the polarization of a beam of particles with spin unity will be discussed. In this case the polarization matrix is much more complicated than the corresponding matrix (1) because it contains a vector and a tensor. In terms of this vector and tensor certain parameters will be defined which provide an estimate for the degree of polarization of the beam. Upper and lower bounds for these parameters will be derived and it will be shown that the vector either has a small magnitude or tends to coincide with one of the principal axes of the tensor.

(*) On leave from the University, Bloemfontein, South Africa.

[1] R. H. Dalitz: *Proc. Phys. Soc.*, A **65**, 175 (1952); W. Lakin: *Phys. Rev.*, **98**, 139 (1955).

For a low degree of polarization either the vector or the tensor term or both can be zero while for a high degree of polarization the tensor term must be present while the vector can again be zero. Described crudely the tensor term can be said to indicate the presence of transverse and longitudinal (with respect to the axis of the tensor coinciding with the vector) polarization while the vector term indicates the presence of circular polarization.

Certain important special cases of the general polarization matrix will be discussed, in particular the case where it takes the form (1) which is applicable to electromagnetic radiation.

2. – General discussion of the polarization matrix.

The density or polarization matrix ϱ for a beam of particles having the same momentum and spin s can be expressed in terms of tensors of rank 0, 1, 2, ..., $2s$ in spin space. The case where $s=\frac{1}{2}$ is particularly simple and ϱ can be generated by $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ in the following way

$$\varrho=\left\{\exp\left[\boldsymbol{\sigma}\cdot\boldsymbol{n})\theta\right]\right\}/\mathrm{Tr}\left\{\exp\left[(\boldsymbol{\sigma}\cdot\boldsymbol{n})\theta\right]\right\}=\tfrac{1}{2}[1+\boldsymbol{\sigma}\cdot\boldsymbol{p}]\,,\tag{1}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices, $\boldsymbol{n}$ is a unit vector and the polarization of the beam is given by the vector

$$\boldsymbol{p}=\boldsymbol{n}\,\mathrm{tgh}\,\theta\,,\qquad \boldsymbol{p}\cdot\boldsymbol{p}\leqslant 1\,,$$

where θ is a real parameter.

In the case where the spin is unity the most general 3×3 polarization matrix must have the form

$$\varrho=(1+\boldsymbol{s}\cdot\boldsymbol{p}+T)/3\,,\tag{2}$$

where $\boldsymbol{p}$ is a vector and the 3×3 matrix $\boldsymbol{s}\cdot\boldsymbol{p}$ has been defined earlier as [2]

$$\boldsymbol{s}\cdot\boldsymbol{p}=\begin{pmatrix}0 & -ip_3 & ip_2\\ ip_3 & 0 & -ip_1\\ -ip_2 & ip_1 & 0\end{pmatrix}$$

while T is a symmetric matrix. The matrix (2) has been constructed in such a way that the required normalization

$$\mathrm{Tr}\,\varrho=1\quad\text{is obtained by choosing}\quad \mathrm{Tr}\,T=0\,.\tag{3}$$

[2] C. B. VAN WYK: *Nuovo Cimento*, **6**, 522 (1957).

If $\boldsymbol{n}_j$ and λ_j are the eigenvectors and eigenvalues of T satisfying

$$(4) \qquad T\boldsymbol{n}_j = \lambda_j \boldsymbol{n}_j \,,$$

where λ_j are real numbers and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ are mutually orthogonal unit vectors, then

$$(5) \qquad T = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 \,, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0,$$

where the projection operators E_j are defined by

$$(6) \qquad E_1 = 1 - (\boldsymbol{s}\cdot\boldsymbol{n}_1)^2 = |\boldsymbol{n}_1\rangle\langle\boldsymbol{n}_1| \quad \text{etc.}$$

and satisfy

$$(7) \qquad E_j\, E_k = \begin{cases} E_j\,, & j = k, \\ 0\,, & j \neq k, \end{cases} \qquad \mathrm{Tr}\, E_j = 1, \quad E_1 + E_2 + E_3 = 1.$$

Because the matrices are linearly dependent, (5) can be written as

$$T = ae_1 + be_3\,,$$

where

$$(8) \qquad \begin{cases} 6^{\frac{1}{2}}a = \lambda_1 - \lambda_2\,, & 2^{\frac{1}{2}}b = \lambda_3\,, \\ (2/3)^{\frac{1}{2}}e_1 = E_1 - E_2\,, & 2^{\frac{1}{2}}e_3 = 3E_3 - 1\,, \end{cases}$$

whence

$$(9) \qquad \mathrm{Tr}\, e_j = 0\,, \qquad \mathrm{Tr}\,(e_j e_k) = 3\,\delta_{jk}\,, \qquad \mathrm{Tr}\,(s_j E) = 0\,,$$

where E is any symmetric matrix. Combination of the above form for T with (2) yields

$$(10) \qquad \varrho = (1 + \boldsymbol{s}\cdot\boldsymbol{p} + ae_1 + be_3)/3\,,$$

for the spin unity polarization matrix. This matrix contains in general eight independent parameters, five are required to specify T (a, b plus three for the orientation of $\boldsymbol{n}_j$) and three to specify $\boldsymbol{p}$.

The form (10) for the polarization matrix is convenient when applied to scattering problems because it is a linear combination of matrices with the trace properties (9). If $\boldsymbol{s}$ is replaced by $(2/3)^{\frac{1}{2}}\boldsymbol{s}$ and $\boldsymbol{p}$ by $(3/2)^{\frac{1}{2}}\boldsymbol{p}$, which will not be done in this paper, then s_j, like e_j, will be normalized to satisfy

$$\mathrm{Tr}\,(s_j s_k) = 3\,\delta_{jk}\,.$$

In the representation in which T is diagonal, the matrix

$$A = \boldsymbol{s}\cdot\boldsymbol{p} + T \tag{11}$$

with the eigenvalues Λ_j has the characteristic equation

$$\det(A - \Lambda) = \Lambda^3 - c\Lambda - \det A = 0\,, \tag{12}$$

where

$$2c = \operatorname{Tr} A^2 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2p^2\,, \tag{13}$$

$$\det A = (1/3)\operatorname{Tr} A^3 = \Lambda_1\Lambda_2\Lambda_3 = \lambda_1\lambda_2\lambda_3 - \lambda_1 p_1^2 - \lambda_2 p_2^2 - \lambda_3 p_3^2\,, \tag{14}$$

where use has been made of $\operatorname{Tr} A = 0$.

Since the eigenvalues of ϱ must lie in the interval $(0, 1)$ and $\operatorname{Tr}\varrho = 1$, it follows from (2) and (11) that

$$\operatorname{Tr}\varrho^2 \leqslant 1 \qquad \text{and} \qquad (1 + \Lambda_1)(1 + \Lambda_2)(1 + \Lambda_3) \geqslant 0\,. \tag{15}$$

Define q by

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2p^2 = 6q^2\,, \qquad (0 \leqslant q \leqslant 1), \tag{16}$$

where the limits for q follow from the first part of (15) while the second part can be written as

$$\det A \geqslant 3q^2 - 1\,. \tag{17}$$

Provided that

$$\det A \leqslant 2q^3\,, \tag{18}$$

the cubic equation (12) has the real roots

$$2q\cos\varphi\,, \qquad -2q\cos(60^\circ - \varphi)\,, \qquad -2q\cos(60^\circ + \varphi)\,, \tag{19}$$

where

$$\cos 3\varphi = (\det A)/(2q^3)\,. \tag{20}$$

If $2q \leqslant 1$ then $-\Lambda_j$ are solutions of (12) that satisfy (17) if Λ_j are such solutions. Hence in this case (17) and (18) can be replaced by

$$|\det A| \leqslant 2q^3\,. \tag{21}$$

The degree of polarization of a beam is determined by Λ_j and therefore through (12) by $\operatorname{Tr} A^2 = 6q^2$ and $\operatorname{Tr} A^3 = 3\det A$. Since (17) and (18) or al-

ternatively (21), define quite narrow limits for $\det A$ once q has been fixed, q can be regarded as a good estimate for the degree of polarization of the beam. Indeed it follows from (17) and (18) that $q = 1$ implies $\det A = 2$, while $q = 0$ implies $\det A = 0$. If $q = 1$ the equality in (18) holds and the eigenvalues of A are $2, -1, -1$ so that ϱ is a projection operator describing complete polarization. Similarly, if $q = 0$ the eigenvalues of A vanish by virtue of (19). Hence A itself vanishes and ϱ is proportional to the unit matrix which indicates that the beam is completely unpolarized.

If $3q^2 \leqslant 1$ then, according to (17) and (14), $\det A$ and T can vanish which means that the polarization matrix need only have the form

$$\varrho = (1 + \boldsymbol{s}\cdot\boldsymbol{p})/3 . \tag{22}$$

Since $\boldsymbol{s}\cdot\boldsymbol{p}$ has the eigenvalue $-p$, this form will lead to negative eigenvalues for ϱ if $p > 1$. Hence if $3q^2 > 1$ the matrix T must contribute to ϱ. On the other hand $q = 1$ is possible even though $p = 0$.

It is possible to draw fairly general conclusions about the relative directions of $\boldsymbol{p}$ and the eigenvectors of T. Suppose that

$$p_1^2 = p_2^2 = p_3^2 , \tag{23}$$

then since $\operatorname{Tr} T = 0$, (14) becomes

$$\det A = \lambda_1\lambda_2\lambda_3 \geqslant 3q^2 - 1 . \tag{24}$$

With Q defined by

$$6Q^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 , \tag{25}$$

(24) can be written, in analogy with (20),

$$\lambda_1\lambda_2\lambda_3 = 2Q^3 \cos 3\alpha \geqslant 3q^2 - 1 \tag{26}$$

while (16) becomes

$$3q^2 = 3Q^2 + p^2 . \tag{27}$$

To keep Q as small as possible and therefore p as large as possible for given q, choose $\alpha = 0$ whence elimination of Q between (26) and (27) yields

$$p^2/3 \leqslant q^2 - (3q^2/2 - \tfrac{1}{2})^{\frac{2}{3}} . \tag{28}$$

Hence if $q^2 = 1 - \varepsilon$, where ε is small, then

$$p^2 \leqslant 3\varepsilon^2/4 . \tag{29}$$

It must therefore be concluded from (23) and (29) that if $q \approx 1$ then the magnitude of $\boldsymbol{p}$ can only be appreciable if its direction shows a preference for one of the principal axes of T.

Next suppose that the λ_j are small while p^2 is not small and p_3^2 is the dominant term. Combination of (14) with (18) then yields

$$2q^3 \geqslant -\lambda_1 p_1^2 - \lambda_2 p_2^2 - \lambda_3 p_3^2 = -\lambda_j R_j\,, \tag{30}$$

where

$$R_j = p_j^2 - p^2/3 \quad \text{and} \quad R_1 + R_2 + R_3 = 0\,, \quad R_1^2 + R_2^2 + R_3^2 = 6R^2\,.$$

These equations are satisfied by

$$R_1 = -2R\cos(60^\circ - \beta)\,, \quad R_2 = -2R\cos(60^\circ + \beta)\,, \quad R_3 = 2R\cos\beta\,,$$

where β is a suitable real parameter. Also

$$\lambda_1 = 2Q\cos(60^\circ - \alpha)\,, \quad \lambda_2 = 2Q\cos(60^\circ + \alpha)\,, \quad \lambda_3 = -2Q\cos\alpha\,,$$

whence

$$2q^3 \geqslant -\lambda_j R_j = 6QR\cos(\alpha - \beta)\,. \tag{31}$$

Since p_3 is the dominant term of p, β is small so that for given Q, R and β, the right hand side of (31) and hence the lower bound for q^3, will be a maximum if α is also small. Thus the polarization is a maximum for given $\boldsymbol{p}$ and (small) Q if $\boldsymbol{p}$ coincides with a principal axis of T, more specifically, $\boldsymbol{p}$ must coincide with the eigenvector of T belonging to a negative eigenvalue of T.

3. – Discussion of special cases.

The general discussion given above will now be illustrated by means of a few important special cases.

Case (*a*): The matrix (2) has its simplest structure when it is a projection matrix describing a completely polarized beam. Equating the symmetric and skew-symmetric parts of the two sides of the equation $\varrho^2 = \varrho$, it follows that

$$T^2 + (\boldsymbol{s}\cdot\boldsymbol{p})^2 = 2 + T\,, \tag{32}$$

$$(\boldsymbol{s}\cdot\boldsymbol{p})T + T(\boldsymbol{s}\cdot\boldsymbol{p}) = \boldsymbol{s}\cdot\boldsymbol{p}\,. \tag{33}$$

The trace of (32) is

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2p^2 = 6\,, \tag{34}$$

which is identical with (16) since $q=1$. From (33) and $(\boldsymbol{s}\cdot\boldsymbol{p})\boldsymbol{p}=0$ follows that

$$T\boldsymbol{p} = -\boldsymbol{p}\,, \qquad (p \neq 0). \tag{35}$$

Hence $\boldsymbol{p}$ is a principal axis of T. Suppose that $\boldsymbol{p}$ coincides with $\boldsymbol{n}_3$ of (4) then

$$\lambda_3 = -1\,. \tag{36}$$

From Tr $T=0$ and (36) follows that (34) can be written

$$(\lambda_1-\lambda_2)^2+4p^2=9\,. \tag{37}$$

Define θ by

$$2p = 3\sin 2\theta\,, \tag{38}$$

then

$$\lambda_1 = 3\cos^2\theta - 1\,, \qquad \lambda_2 = 3\sin^2\theta - 1\,. \tag{39}$$

Hence ϱ contains only four independent parameters, θ and three more to specify the orientation of $\boldsymbol{n}_j$. The eigenfunctions of ϱ satisfy

$$\varrho\boldsymbol{m}_1 = \boldsymbol{m}_1\,, \qquad \varrho\boldsymbol{m}_2 = 0\,, \qquad \varrho\boldsymbol{m}_3 = 0\,, \tag{40}$$

where

$$\boldsymbol{m}_j = (\exp\,[it\theta])_{jk}\boldsymbol{n}_k \tag{41}$$

and

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t^3 = t\,. \tag{42}$$

Case (*b*): A slightly more complicated case is obtained by replacing the condition (34) by (16) with $q<1$. If Λ_1, Λ_2, $\Lambda_3=\lambda_3=-1$ are the eigenvalues of A while θ is defined by

$$\operatorname{tg} 2\theta = 2p/(\lambda_1-\lambda_2)\,, \qquad (\lambda_1 \geqslant \lambda_2,\ |\theta| \leqslant 45°) \tag{43}$$

then

$$\lambda_1-\lambda_2 = (\Lambda_1-\Lambda_2)\cos 2\theta\,, \tag{44}$$

$$2p = (\Lambda_1-\Lambda_2)\sin 2\theta\,, \tag{45}$$

$$\lambda_1+\lambda_2 = \Lambda_1+\Lambda_2\,, \tag{46}$$

$$(\Lambda_1-\Lambda_2)^2 = 12q^2-3\,. \tag{47}$$

In this case ϱ contains the additional parameter p. In terms of θ and the eigenfunctions of T, the eigenfunctions of ϱ are again given by (41) and (42).

Case (*c*): Next allow $q < 1$, $\lambda_3 > -1$ while still requiring $\boldsymbol{p}$ to coincide with $\boldsymbol{n}_3$. The polarization matrix now contains six independent parameters. Formulae analogous to (43)–(46) are valid in this case and the eigenfunctions of ϱ are again given by (41) and (42).

4. – Polarization matrix for a photon beam.

The photon is a particle with spin unity and if its momentum has the direction $\boldsymbol{n}_3$ then its spin states are the eigenfunctions of the operator $\boldsymbol{s}\cdot\boldsymbol{n}_3$. If $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ is a set of orthonormal vectors then

$$(\boldsymbol{s}\cdot\boldsymbol{n}_3)(\boldsymbol{n}_1 \pm i\boldsymbol{n}_2) = \pm(\boldsymbol{n}_1 \pm i\boldsymbol{n}_2) \tag{48}$$

and these two eigenstates of $\boldsymbol{s}\cdot\boldsymbol{n}_3$ correspond to the two states of circular polarization of a photon beam with direction $\boldsymbol{n}_3$. As a result of the transversality of electromagnetic radiation, any other state of polarization can be expressed as a linear combination of these two states. This means that $\boldsymbol{n}_3$, the third eigenstate of $\boldsymbol{s}\cdot\boldsymbol{n}_3$, regarded as an eigenstate of photon polarization, is always empty. Therefore if the above formalism for particles with spin unity is to be applied to a photon beam, one of the eigenvalues of ϱ must vanish as in the cases (*a*) and (*b*). Then ϱ must reduce to the 2×2 matrix (1). This Stokes parameter form of ϱ has been applied extensively to problems involving electromagnetic radiation [3].

To deduce this form for the photon polarization matrix from the above formalism, it is convenient to choose the representation in which T is diagonal with the 3-axis along $\boldsymbol{n}_3$. Then it follows from (2), (4) and Cases (*a*) and (*b*) that

$$\varrho = \frac{1}{2}\begin{pmatrix} 1+P_3 & P_1 - iP_2 & 0 \\ P_1 + iP_2 & 1 - P_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{49}$$

where

$$P_1 = 0\,, \qquad 3P_2 = 2p\,, \qquad 3P_3 = \lambda_1 - \lambda_2\,. \tag{50}$$

Alternatively, we need only choose the 3-axis along $\boldsymbol{n}_3$ then $P_1 \neq 0$ and T will

[3] See for example the article by S. R. DE GROOT and H. A. TOLHOEK in *Beta and Gamma-ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955).

not necessarily be diagonal but ϱ will still have the 2×2 form of (49) which is essentially the formula (1). From (44), (45) and (47) follows that

$$9P^2 = 9(P_1^2 + P_2^2 + P_3^2) = 12q^2 - 3\,.$$

For complete polarization $q=1$ and $P=1$ while if $q<1$ then $P<1$.

From (50) it is clear that $\boldsymbol{P}$ cannot be regarded as a vector in ordinary space.

It will now be shown that the polarization matrix for a completely polarized beam of spin unity particles has a structure analogous to the the first part of (1). Define

$$\eta = \begin{pmatrix} k_3 & -ik_2 & 0 \\ ik_2 & -k_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which satisfies $\eta^3=\eta$ and hence has the eigenvalues 0, ±1 if $k_2^2+k_3^2=1$. The matrix

$$\varrho = (\exp[\eta\theta])/\{\mathrm{Tr}\,(\exp[\eta\theta])\} = (1 + s_3 p + T)/3\,,$$

where

$$p = (3k_2\sinh\theta)/(1+2\cosh\theta)$$

and the eigenvalues λ_j of T satisfy

$$\lambda_1 - \lambda_2 = (6k_3\sinh\theta)/(1+2\cosh\theta)\,,\qquad \lambda_3 = -2\,(\cosh\theta-1)/(1+2\cosh\theta)\,.$$

In the limit as $\theta\to\infty$ the matrix ϱ is exactly the same as for case (a).

5. – Types of polarization

The three independent parameters a, b and p of the matrix (10) define q by

$$3a^2 + 3b^2 + 2p^2 = 6q^2$$

and indicate to what extent the three operators e_1, e_3 and $\boldsymbol{s}\cdot\boldsymbol{p}$ respectively are present in ϱ. Their effect on the type of polarization can be seen as follows. If in (41), $\theta=0$ and hence $p=0$, then ϱ and e_1 have the same real eigenfunctions. If $\lambda_3=-1$, the corresponding state of ϱ is empty and the polarization of the beam is linear, transverse with respect to the 3-axis. If $\lambda_3>-1$ there will be a longitudinal component of polarization. However

if $\theta = 45°$ then $a = 0$ and ϱ and $\boldsymbol{s}\cdot\boldsymbol{p}$ have the same eigenfunctions (48) which indicate circular polarization. Hence the magnitudes of a, b and p can be regarded, somewhat crudely, as measures of the linear transverse, longitudinal and circular polarization of the beam.

* * *

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RIASSUNTO (*)

Si discute la matrice 3×3 che descrive la polarizzazione di un fascio di particelle con spin 1. Si definisce il grado di polarizzazione e si discute il tipo di polarizzazione in termini del vettore e del tensore facenti parte della matrice. Si dimostra che il vettore tende a coincidere con uno degli assi principali del tensore. Si indica l'equivalenza del caso speciale adatto col trattamento del parametro di Stokes della polarizzazione della radiazione elettromagnetica.

(*) *Traduzione a cura della Redazione.*

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Spectrum of Protons from n, p Reactions at 13.4 and 17.5 MeV.

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(ricevuto il 14 Aprile 1958)

Summary. — In this paper measurements are presented on the energy distribution of protons emitted in n, p reactions on Al and Ni in the natural isotopic mixture. These measurements have been taken with neutrons of energy of 17.5 and 13.4 MeV, to the purpose of studying how the proton energy distribution varies with the incoming neutron energy, in order to establish which process is responsible for the n, p reaction. The results give evidence to the fact that the statistical evaporation theory is insuitable for explaining the proton spectra obtained, and that the spectrum shape does not depend on the residual nucleus excitation energy but seems more determined by the energy of emitted protons.

1. – Introduction.

In a previous paper (1), evidence has been put forward, that the energy distribution of protons emitted in n, p and p, p′ nuclear reactions does not depend on the excitation energy of the residual nucleus of the reaction.

From the comparison of the various spectra, it was concluded that the energy distribution seemed rather to be determined by the energy of the

(1) L. Colli, U. Facchini and S. Micheletti: *Nuovo Cimento*, **5**, 502 (1957).

emitted proton. In other words, spectra of protons corresponding to different excitation energies of the residual nucleus were practically the same. Since this fact is in disagreement with the evaporation theory in its most classical form, according to which the spectrum is determined by the level density of the residual nucleus, we felt, it would be interesting to repeat the measurements of the energy distribution of the protons emitted by n, p reactions for two different energies of the incoming neutrons, so as to obtain spectra that can be directly compared with each other. These measurements were also undertaken for the more general purpose of studying how the proton spectra varied with the energy of the incoming neutrons, so as to compare them with the other theories that take into account different mechanisms of reactions, such as the « direct effect ».

In this paper we are presenting these measurements, obtained with the two neutron energies, 13.4 and 17.5 MeV, and with the reaction Al(n, p)Mg and Ni(n, p)Co.

2. – Experimental apparatus.

The neutron source for these measurements is obtained with the 1.5 MeV Van de Graaff accelerator of Zürich University and by means of the reaction ${}^{2}D_1 + {}^{3}T_1 = n_0^1 + \alpha_2^4$.

By bombarding a target of zirconium and tritium (thin compared with the range of 1.5 MeV deuterons) it is in fact possible to obtain neutrons of 17.5 MeV energy emitted at 0° with respect to the direction of the incoming deuterons, and at 130° of 13.4 MeV energy. With a current of deuterons on the target of 20 μA, a flux was obtained of about 10^8 neutrons/s in the total solid angle.

The protons emitted in the n, p reaction are measured with the same detector used in some of our previous works [2-4], which consists of a combination of two proportional counters and a scintillation spectrometer in coincidence with each other. After the triple coincidence, the pulses are sent to a 100-channel pulse-height analyzer.

The target, in which the n, p reaction is to take place, is placed at 7.5 cm from the neutron source and has a diameter of 4 cm.

Using a thin layer of polythene (19.6 mg/cm²) of 1 cm diameter as a target, recoil protons are obtained, which in this geometry are sufficiently mono-

(2) C. BADONI, L. COLLI and U. FACCHINI: *Nuovo Cimento*, **4**, 1618 (1956).

(3) L. COLLI and U. FACCHINI: *Nuovo Cimento*, **5**, 309 (1957).

(4) L. COLLI, U. FACCHINI, I. IORI, G. MARCAZZAN, A. SONA and M. PIGNANELLI: *Nuovo Cimento*, **7**, 400 (1958).

chromatic to furnish good lines for the calibration of the energy scale. Fig. 1 shows such lines as obtained for the two positions of the proton detector, corresponding therefore to neutrons of 17.5 and 13.4 MeV respectively. These

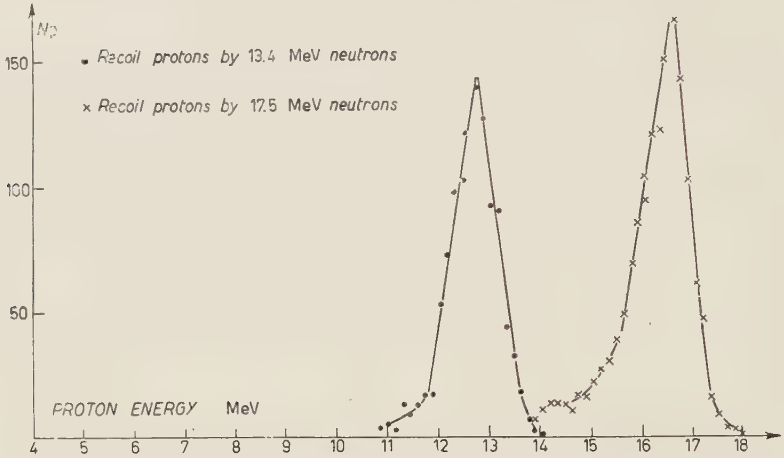

Fig. 1. – Calibration spectrum of recoil protons obtained from a polithene target at 17.5 and 13. MeV neutron energies.

spectra of recoil protons of hydrogen give also the energy resolution of the detector. It is noteworthy that a fair part of the line width is due to the angular dispersion of the geometrical arrangement.

With this experimental lay-out we have measured the spectra of the protons emitted by an Al and a Ni target, using incident neutrons of the two energies.

Every measurement consists in detecting successively the spectrum of recoil protons of hydrogen, the background spectrum, and that of the element under study.

The geometrical arrangement is made in such a way that the protons emitted by Al and Ni are accepted by the coincidence system only if emitted with an angle between 0° and 35°, 18° being the most favourable angle.

4. – Results.

The results of these measurements are shown in Fig. 2 and 3. Fig. 2 shows the spectrum of the protons emitted in the reaction Al(n, p) for the two energies of neutrons used, together with the spectrum obtained at 14.7 MeV at Milan ([5]).

([5]) This measurement was conducted with an improved type of detector with respect to the previous ones which will be published shortly by G. Marcazzan, M. Pignanelli and Anna Sona.

The spectra were drawn superposed for easy comparison. The proton energy in the abscissa refers to the center of mass system. The scale of the

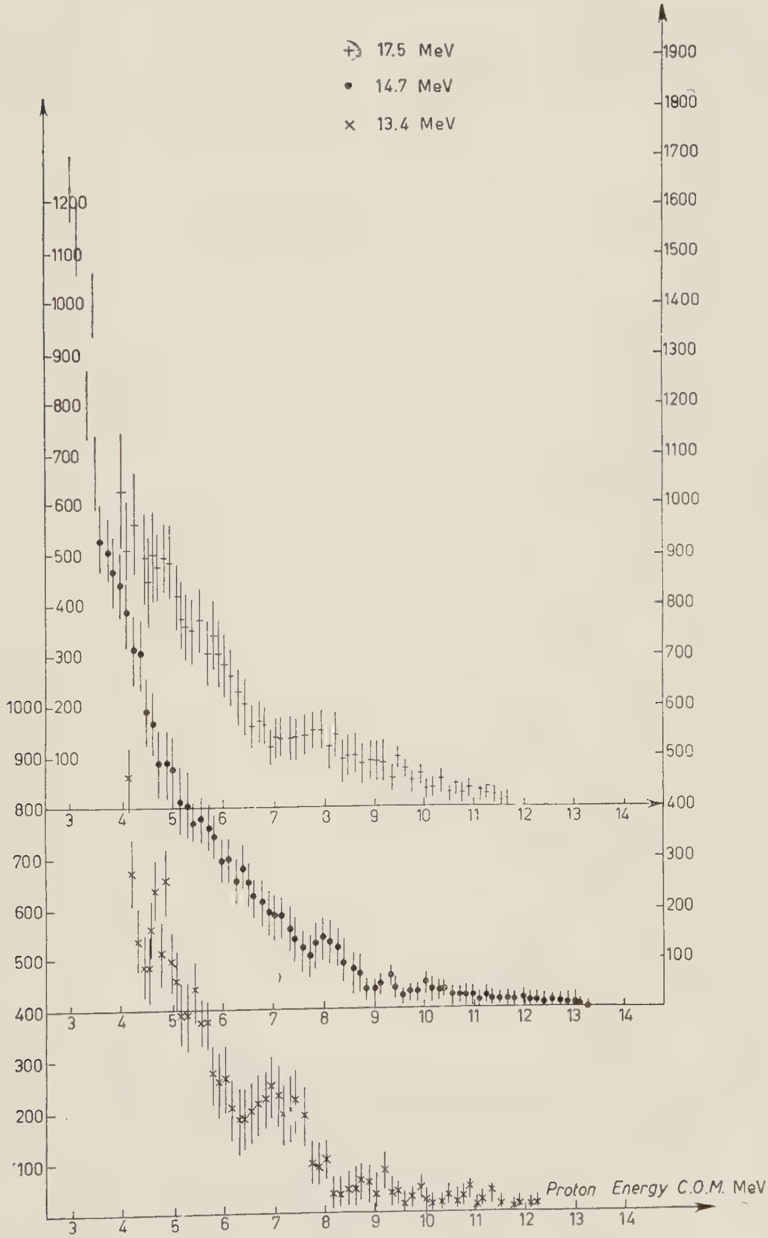

Fig. 2. – Spectra of protons obtained from the reaction Al(n, p)Mg at 13.4, 14.7 and 17.5 MeV neutron energies.

ordinate in this case is arbitrary and the spectra have been normalized with respect to each other, at 5 MeV.

The spectrum obtained at 13.4 MeV shows clearly a peak at 7 MeV, that at 14.7 shows a similar peak at 8 MeV, and finally, the spectrum obtained at 17.5 MeV has the same form as the previous ones at the low energy side, shows a large peak at 8 MeV and is richer than the previous ones above (8 ÷ 9) MeV.

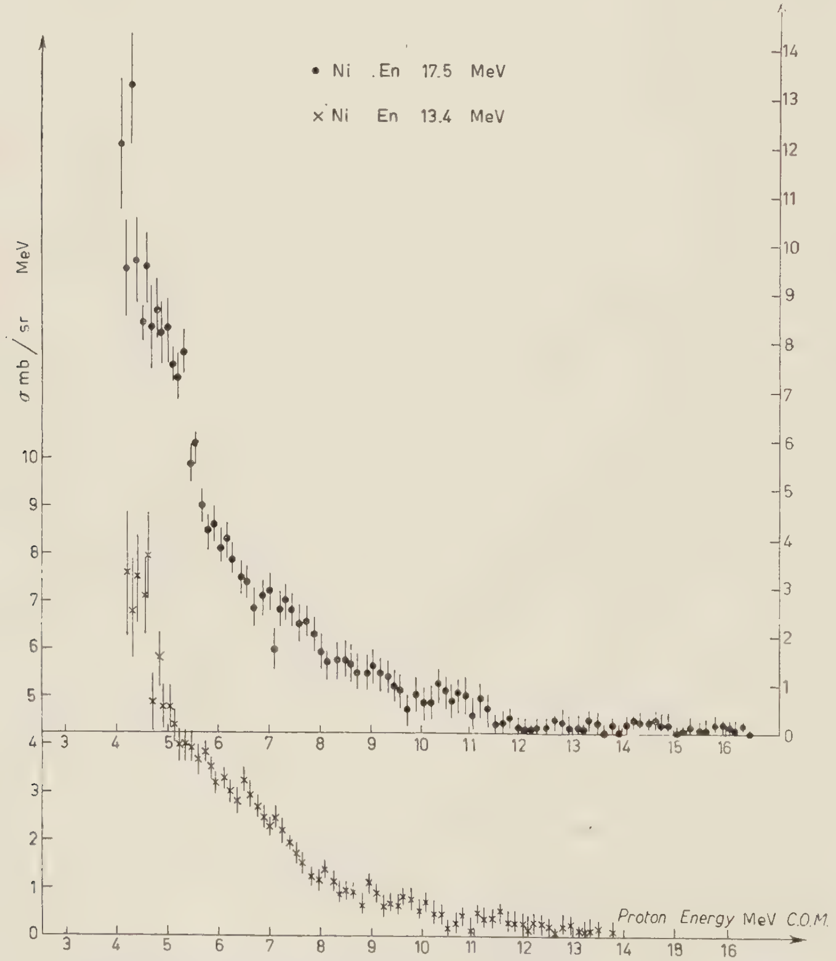

Fig. 3. – Spectra of protons from Ni(n, p) Co reaction at 13.4 and 17.5MeV neutron energies.

The result obtained for the spectrum of protons from Ni (Fig. 3) is similar. The two spectra are very similar to each other. The ordinates give the cross-section in mb/sr MeV, corresponding to the useful angle detected. The spectrum at 17.5 MeV shows greater intensity and a tail at the highest energies.

4. – Discussion.

The proton spectrum emitted in n, p reaction of Ni has been studied by many authors in recent years (6,7).

Its angular distribution has also been studied (1,7). All the existing measurements, as a whole, lead to the conclusion that this reaction cannot be interpreted on the basis of the statistical evaporation theory for the following reasons:

1) The angular distribution is slightly anisotropic It is interesting to note that the shape of the spectrum is about the same at all angles.

2) If, on the basis of the evaporation theory, the level density of the residual nucleus, ω, is calculated from the spectrum, by dividing it by $\varepsilon\sigma_c$, where ε is the energy of the emitted proton and σ_c the cross-section for the reverse reaction, the curve obtained $\omega(E)$ (E = excitation energy of the residual nucleus) is not in accordance with the one foreseen by the fundamental hypothesis that the nucleons in the nucleus can be described with a Fermi gas, as has been shown in a previous work (1).

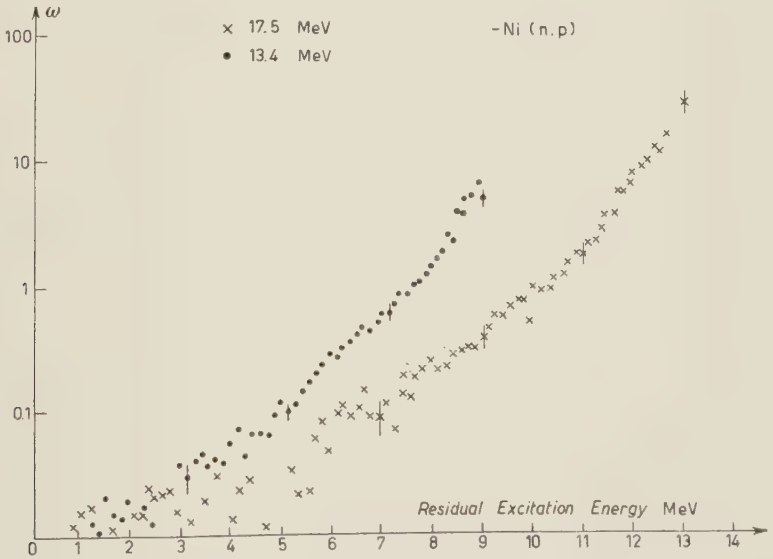

Fig. 4. – Curves $\omega(E)$ obtained dividing the curves of Fig. 3 for $\sum \sigma_c$, and supposed to represent, in the hypothesis of evaporation theory the density of level of residual nucleus. The ordinates are given in arbitrary units.

(6) D. L. ALLAN: *Proc. Phys. Soc.*, **70**, 195 (1957).
(7) P. V. MARCH: private communication.

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Furthermore, the curves $\omega(E)$ that can be obtained from our measurements at various energies do not agree among themselves, as shown in Fig. 4. This result confirms what has been demonstrated in ([1]), where $\omega(E)$ derived from (n, p) and (p, p′) were compared.

All the $\omega(E)$ derived from reactions n, p show a steeper rise towards high excitation energies than expected, and this is due to the fact that the spectra are rich in low energy protons.

The measurements reported here show that the deviation in the spectra at low energies cannot be attributed to the contribution of protons from n, np reactions as has been suggested by some authors in similar cases ([6,8]). In fact, on the basis of the evaporation theory calculated by BLATT and WEISSKOPF ([9]), the ratio between the cross-section of n, np and n, p increases rapidly with increasing energy available for the reaction above the threshold.

Since the n, np threshold of Ni is 6 MeV for ^{60}Ni and 7.6 for ^{58}Ni a change of energy from 13.4 to 17.5 for the incoming neutron represents an appreciable change in the energy available for the reaction.

The spectrum obtained at 17.5 MeV should show a greater contribution of protons obtained from n, np reaction, *i.e.* the steep part at the low energy side should be much more important and should begin at much higher proton energy due to the fact that in this spectrum we have the n, np threshold at about 10 MeV proton energy instead of at about 6 MeV, as in the case of the 13.4 MeV neutron energy spectrum. But the measurements show spectra of the same shape.

Confirmation has therefore been obtained that the so-called $\omega(E)$ curves do not represent density of levels, the n, p reaction of Ni does not develop along the statistical evaporation process only, and that at least half of the spectra emitted forward above proton energy of 4 MeV are due to a different process. Moreover, the fact that the spectra obtained at all angles have similar shape suggests that the greater contribution to the reaction at all angles of proton emission is due to the same process.

It has been suggested by various authors that, in these reactions, a process is present of the type of direct interaction between the neutron and one nucleon of the nucleus. If the protons of n, p reactions are to be attributed to this process, then one necessarily thinks that a reflexion phenomenon also takes place inside the nucleus in order to explain the protons emitted backward.

The measurements reported here confirm directly what has been stated in ([1]), *i.e.*, that the spectrum is not determined by the excitation energy of the residual nucleus, and they show clearly that the spectrum seems to be

([8]) S. HAYAKAWA, M. KAWAI and K. KIKUCHI: *Progr. Theor. Phys.*, **13**, 415 (1955).
([9]) J. M. BLATT and V. F. WEISSKOPF: *Theoretical Nuclear Physics* (New York, 1952).

determined by the energy of the outgoing protons itself, since a change in the incoming energy of 4 MeV does not alter the shape of the spectrum except for high energy protons, which evidently can be emitted.

The case of Al(n, p) is similar. The angular distribution for this spectrum has been measured by BROWN *et al.* ([10]), and has been found to be anisotropic even at energies lower than 4 MeV.

The fact that the spectra obtained at 13.4, 14.7 and 17.5 MeV have the same shape at low energy, excludes, even in this case, a great contribution of n, np reaction which, for Al, has a threshold value of 8.3 MeV.

It is interesting to note the existence of the peak, which, in the spectrum obtained at 13.4 MeV is found at 7 MeV proton energy, in that obtained

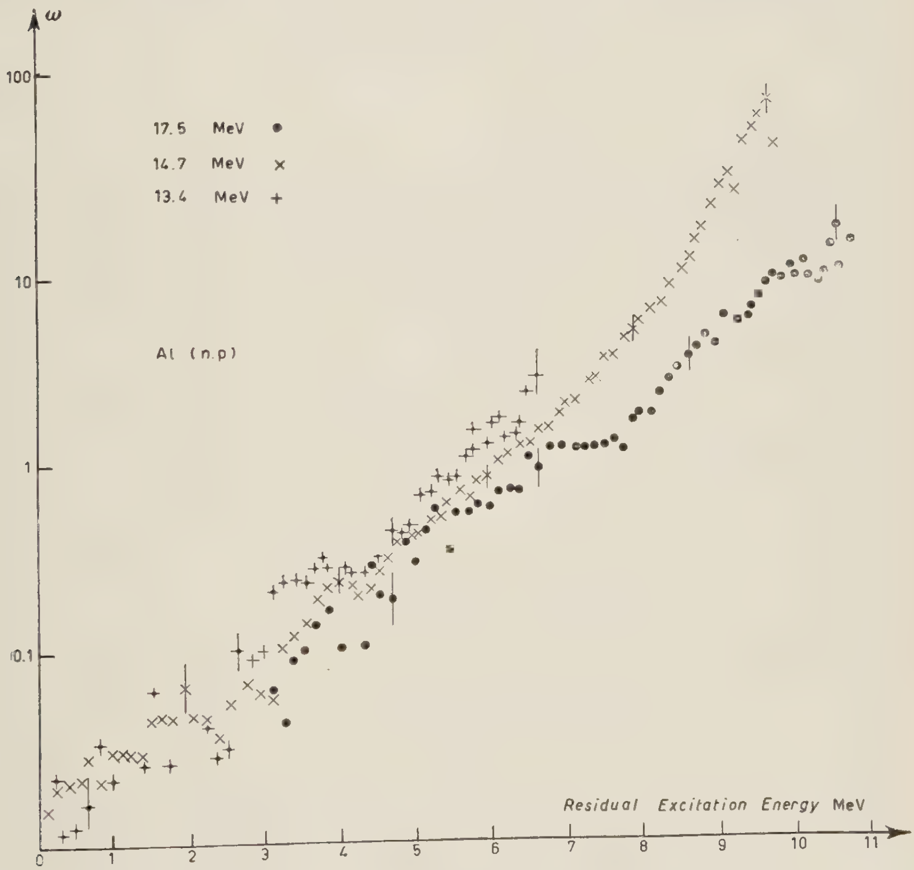

Fig. 5. – Curves $\omega(E)$ calculated from the proton spectra of aluminium, as given in Fig. 2.

([10]) G. BROWN, G. C. MORRISON, H. MUIRHEAD and W. T. MORTON: *Phil. Mag.*, 2, 785 (1957).

at 14,7 at 8 MeV proton energy, and in the spectrum at 17.5 MeV is found again at 8 MeV proton energy.

It would be very interesting to establish if the position of this large peak at 8 MeV depends on proton energy or on residual excitation energy. In the first case, it could be related to an effect due to the existence of standing waves for the outgoing proton in the nuclear potential, such as it is foreseen by the optical model theory.

The results for the two spectra taken at 14 and 17,5 MeV seem to agree with this explanation, but the spectrum at 1,35 MeV shows the bump at lower energy.

On the whole, these data show clearly that in the case of aluminium, too, the n, p reaction proceeds, at least for its bigger part, along a different process than that of evaporation, with the same characteristics as in the case of nickel (Fig. 5).

* * *

The authors have the honour to thank Prof. HANS H. STAUB for the kind hospitality shown to two of them and for his constant assistance with discussion and advice, Prof. U. FACCHINI for discussions and assistance in the measurements, Prof. C. SALVETTI for permitting the collaboration of the Istituto di Fisica of Milan, Prof. B. FERRETTI for the discussions that led to the undertaking of these measurements, Drs. A. M. LANE and D. L. ALLAN for the discussion of the results, Prof. M. SILVESTRI for the preparation of the gaseous deuterium, Dr. G. MARCAZZAN for the assistance in the measurements, and Dr. G. SCHIFFRER for some discussions on theoretical interpretations.

RIASSUNTO

In questo lavoro vengono presentate misure della distribuzione energetica dei protoni emessi nella reazione n, p sugli elementi alluminio e nichel nella miscela isotopica naturale. Le misure sono state fatte con neutroni dell'energia di 13.4 e di 17.5 MeV, allo scopo di studiare la variazione con l'energia dei neutroni incidenti della distribuzione energetica dei protoni emessi, per stabilire quale meccanismo è responsabile della reazione. I risultati confermano l'inadeguatezza della evaporazione statistica per spiegare anche in parte gli spettri ottenuti, e mettono in evidenza che la forma degli spettri ottenuti non è determinata dall'energia di eccitazione del nucleo residuo ma sembra piuttosto determinata dall'energia stessa del protone emesso.

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Determination of the Circular Polarization of the β-Rays Bremsstrahlung with a New γ-Polarimeter.

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(ricevuto il 16 Aprile 1958)

Summary. — The circular polarization of the bremsstrahlung produced by the ^{32}P β rays has been measured with an analyzer recently proposed by Beard and Rose. The degree of polarization is found to be consistent with the theoretical previsions. The polarimeter is found to work quite satisfactorily.

1. – Increasing interest has been put in the last years in the problem of the detection of the circular polarization of γ-rays, mainly in connection with the new ideas on β-decay.

In order to investigate this problem, we have set up a new γ polarimeter following a suggestion of Beard and Rose and have calibrated it by measuring the circular polarization of the bremsstrahlung beam produced by the ^{32}P β-rays. This choice was made because the longitudinal polarization of these β-rays has been directly measured by using Möller scattering, or Coulomb scattering after deflection of the β-rays in an electric or magnetic field in order to transform their longitudinal polarization in a transverse one (*).

The photon circular polarization analyzers are generally based on the dependence of the Compton cross-section on the polarization state of the incident photon and of the target electron.

(*) During this experiment a measurement of circular polarization of the bremsstrahlung from ^{32}P has been reported by Bohem and Wapstra.

In fact, if the incident beam is circularly polarized and the electrons' spin is taken into account, the differential cross-section may be written [1]:

$$(1)\qquad \frac{\mathrm{d}\sigma(\boldsymbol{\xi}_0,\boldsymbol{\zeta}_0)}{\mathrm{d}\Omega}=\frac{1}{2}r_0^2\frac{k^2}{k_0^2}\left[(1+\cos^2\theta)+(k_0-k)(1-\cos\theta)-\right.$$

$$\left.-P((1-\cos\theta)\boldsymbol{\zeta}_0\times(\boldsymbol{k}_0\cos\theta-\boldsymbol{k})\right],$$

where P is the degree of circular polarization, being $P>0$ in the case of left-hand polarization and $P<0$ in the opposite case; $\boldsymbol{\xi}_0$ and $\boldsymbol{\zeta}_0$ are the polarization vectors of the incident photon and of the electron; $\boldsymbol{k}_0$ and $\boldsymbol{k}$ are the momenta of the incident and scattered photons and θ the scattering angle

The analyzers till now used are based on the measurement of the difference between Compton cross-sections on electrons having their spin parallel or antiparallel to the incident photon momentum $\boldsymbol{k}_0$.

GUNST and PAGE [2-4] have set up an apparatus of this kind based on the measurement of the difference between the total cross-sections in the two extreme conditions mentioned above, *i.e.* on the measurement of the different transmission coefficient of the γ-rays in an iron bar alternatively magnetized in opposite directions.

As it is known, however, this method is not efficient for γ-rays energies about 0.65 MeV, because the total cross-section is not depending on the circular polarization of the photons of such energies.

Another method measures the difference in the differential Compton cross-section at small angles. This method has been used by WHEATLEY and others [5], or more recently, by BOHEM and WAPSTRA [6], who have developed a polarimeter with cylindrical symmetry which greatly improves the intensity.

On the other hand, DE BENEDETTI and others [7] use Compton backscattering in iron magnetized with parallel and antiparallel direction to the incident radiation.

The method, proposed by BEARD and ROSE [8], is based on the measure-

(1) See, for example, also for a complete bibliography: H. A. TOLHOEK: *Rev. Mod. Phys.*, **28**, 277 (1956).

(2) S. B. GUNST and L. A. PAGE: *Phys. Rev.*, **92**, 970 (1953).

(3) G. TRUMPY: *Nuclear Phys.*, **2**, 664 (1956-57).

(4) M. GOLDHABER, L. GRODZINS and A. W. SUNYAR: *Phys. Rev.*, **106**, 826 (1957).

(5) J. C. WHEATLEY, W. J. HUISKAMP, A. N. DIDDENS, M. J. STEENLAND and H. A. TOLHOEK: *Physica*, **21**, 841 (1955).

(6) F. BOHEM and A. H. WAPSTRA: *Phys. Rev.*, **106**, 1364 (1957); **107**, 1202, 1462 (1957); **109**, 546 (1958).

(7) M. BERNARDINI, P. BROVETTO, S. DE BENEDETTI and S. FERRONI: *Nuovo Cimento*, **7**, 416 (1958).

(8) D. B. BEARD and M. E. ROSE: *Phys. Rev.*, **108**, 164 (1957).

ment of the azymuthal asymmetry which appears in the Compton scattering on electrons having the spin perpendicular to the propagation vector of the incident photons.

In such a case, the eq. (1) becomes:

$$\sigma(\theta, \varphi) = \frac{1}{2} r_0^2 \frac{k^2}{k_0^2} [(1 + \cos^2\theta) + (1 - \cos\theta)(k_0 - k) + P(1 - \cos\theta)\zeta_0 k \sin\theta \cos\varphi)] \,,$$

where φ is the angle between the plane $(\boldsymbol{k}, \boldsymbol{k}_0)$ and the plane $(\boldsymbol{\zeta}_0, \boldsymbol{k}_0)$.

Therefore, the azimuthal anisotropy turns out to be:

$$\eta_\theta(\varphi, \pi - \varphi) = 2\,\frac{\sigma(\theta, \varphi) - \sigma(\theta, \pi - \varphi)}{\sigma(\theta, \varphi) + \sigma(\theta, \pi - \varphi)} = 2\,\frac{P(1 - \cos\theta)\,\zeta_0 k \sin\theta \cos\varphi}{(1 + \cos^2\theta) + (k_0 - k)(1 - \cos\theta)} \,.$$

The scattering angle θ, for which η attains its maximum, has been evaluated as a function of the energy by the above mentioned authors. It turns out to be $\geqslant \pi/2$ for incident γ-ray energies $\leqslant m_e c^2$ and decreases with increasing energy; it becomes, for example, about $\pi/4$ for energies in the range of 5 MeV.

Maximum asymmetry is obviously reached between $\varphi = 0$ and $\varphi = \pi$. Under optimum condition of θ and φ, the asymmetry increases steeply with increasing energy up to an absolute maximum of about 5.6%, attainable at 0.51 MeV in a scatterer of saturated iron; then it decreases slowly and at about 10 MeV, it is found to be, for example, $\sim$1.7%.

The order of magnitude of the highest asymmetry observed by means of other polarimeters is the same, at least for the energies which are found in radioactivity.

This method appears especially advantageous when one wants to know with remarkable precision both the polarization state and the original direction of the γ-ray.

Our experimental conditions certainly were not the best ones, both for what concerns the bremsstrahlung γ spectrum characteristics, and for the employed apparatus, which had not been planned on the purpose. Nevertheless, the results point out that such a method is as efficient as the other ones used till now.

2. – A sandwich source, about 100 mC of ^{32}P, has been made using layers of red phosphorus alternated with lead foils of 0.7 mm thickness, in order to reduce the self-absorption of β-rays in phosphorus and increase the intensity of the bremsstrahlung beam in the lead.

The spectrum of pulses produced by the direct beam in a NaI(Tl) crystal (2.5 cm × 2.5 cm) is shown in Fig. 1.

Our experimental setting is schematically shown in Fig. 2.

The thickness l of the Fe scatterer has been chosen in order to obtain the maximum efficiency for energies of the incident γ-rays of about 1 MeV. We have therefore attained the highest probability for an incident photon (perpendicular to the target) to be scattered without multiple scattering into the solid angle, as seen by the counter. This probability is given by

$$\int_0^l \frac{\mathrm{d}\sigma_c(\theta)}{\mathrm{d}\Omega} \exp\left[-\sigma_T x\right] \exp\left[\frac{-\sigma'_T(l-x)}{\cos\theta}\right] \mathrm{d}x\,,$$

where $\mathrm{d}\sigma_c(\theta)/\mathrm{d}\Omega$ is the Compton cross-section, θ is the chosen scattering angle; $\sigma_T = 0.06\ \mathrm{g/cm^2}$ and $\sigma'_T = 0.10\ \mathrm{g/cm^2}$ are the total cross-sections in the iron for the incident and the scattered photon. The best thickeness so found is 0.7 cm.

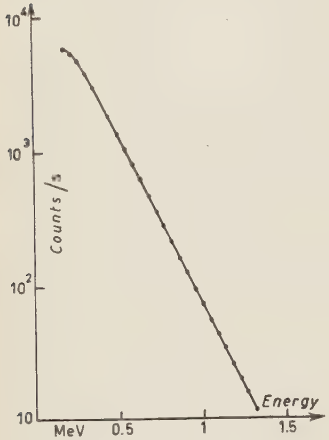

Fig. 1. – Pulse spectrum of the direct beam.

The scattering angle was chosen 80° in order to obtain the maximum asymmetry, according to the results of BEARD and ROSE. We have worked with different pulse height discriminations corresponding to the following energy range of the incident photons: (0.2 ÷ 0.4) MeV; (0.4 ÷ 0.8) MeV; (0.8 ÷ 1.5) MeV.

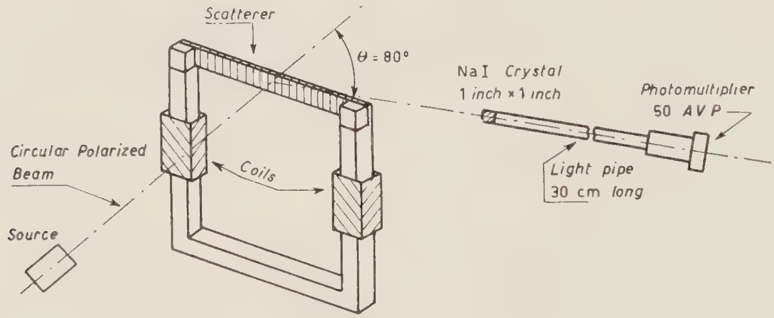

Fig. 2. – Experimental arrangement.

The corresponding foreseen asymmetries have been evaluated assuming a degree of polarization of the β-rays of about $-v/c$ (9), according to the two component neutrino theory, and using the production cross-section of polarized bremsstrahlung evaluated by McVOY and by BÖBEL (10). The polari-

(9) T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957); L. D. LANDAU: *Nucl. Phys.*, **3**, 127 (1957); *Žurn. Ėxp. Theor. Fyz.*, **32**, 405 (1957).

(10) N. M. McVOY: *Phys. Rev.*, **106**, 828 (1957); G. BÖBEL: *Nuovo Cimento*, **6**, 1241 (1957).

meter efficiency has been evaluated from the data of BEARD and ROSE (see reference [8]). The so calculated η_t asymmetries are shown in the last column of the table.

The photomultipliers we have used (Philips 50 AVP) are particularly sensitive to magnetic fields (*); the instrumental asymmetry η(str.), however, has been reduced to negligible values by using a light-pipe 30 cm long and three magnetic shields around the phototubes. Such an asymmetry has been measured by substituting the bremsstrahlung source with a ^{60}Co sample; with the discriminator window at (35÷45) V (see the table), we have found the value

$$\eta\,(\text{str.}) = (0.07 \pm 0.12)\,\%$$

small enough for our requirements.

The experimental asymmetry, due to the polarization state of the γ-rays, is defined by

$$\eta = 2\,\frac{n_1 - n_2}{n_1 + n_2}\,,$$

where n_2 represents the counts produced by the photons scattered in the counter when the magnetic field $\boldsymbol{H}$ is in the direction of the counter and n_1 the counts produced with the magnetic field in the opposite direction. The expression may be written in the useful form:

$$\eta = 2\,\frac{N_1 - N_2}{N_1 + N_2}\cdot\frac{1}{1-\alpha}\,,$$

where N_1 and N_2 have a similar meaning as n_1 and n_2 but are referred to the experimental total counts and α is the ratio of the number of counts without the scatterer to that of counts with the scatterer.

The experimental results are given in the Table I.

TABLE I.

Discrimination window (in volt)	Corresponding range of the incident energy (in MeV)	N_2	N_1	N (s)	$\eta\times 100$	$\eta_t\times 100$ (calculated)
15÷25	0.2÷0.4	88.382	86.920	∼ 40	− 3.3 ± 1.1	− 3.3
25÷35	0.4÷0.8	143.861	142.173	∼ 20	− 2.1 ± 0.7	− 4.1
35÷45	0.8÷1.5	74.529	73.344	∼ 4	− 2.5 ± 0.8	− 3.9
* 35÷45	0.8÷1.5	206.895	204.690	∼ 3	− 2.1 ± 0.6	− 3.9

(*) They offer, however, a good energy resolution, good stability, and low noise.

The measurements have been made inverting the magnetic field with 10 minute intervals. The last measurement, starred in the table, has been made in exceptionally steady experimental conditions and consists of 115 measurements for each one of the two values of the magnetic field.

The distribution of the corresponding 115 η-values for each couple of measurements fully agrees with a gaussian distribution around their medium value, as it is shown by the χ^2 test. The «Student» test, applied to the two series of values for N_1 and N_2 shows that these series are physically distinct with a probability very close to unity.

The η negative value shows that the spin of the polarized photons is antiparallel to the momentum $\boldsymbol{k}_0$. This means that also the spin of the β-rays is antiparallel to their momentum, as it is shown also by preceding experiences (11) and it may be expected in the case of negative electrons.

Each one of the experimental results is, within the errors, consistent with the values predicted by the theory. All of them, however, are smaller than the calculated ones, as it might be expected if any lack of homogeneity in the magnetization of the target and any multiple scattering are taken into account.

Measurements are being made with an experimental set-up which has been improved, as concerning the geometrical arrangement as well as the selection of the events in order to study the dependence of the degree of polarization on the energy of the bremsstrahlung photons.

* * *

We wish to express our deepest gratitude to Prof. E. PANCINI for all helpful suggestions he has constantly given us on this work. We also wish to thank Dr. BÖBEL for some useful discussions.

(11) H. FRAUENFELDER, A. O. HANSON, N. LEVINE, A. ROSSI and G. DE PASQUALI: *Phys. Rev.*, **107**, 643 (1957); H. DE WAARD and O. J. POPPEMA: *Physica*, **23**, 597 (1957); H. J. LIPKIN, S. CUPERMAN, T. ROTHEM and A. DE SHALIT: *Phys. Rev.*, **109**, 223 (1958); F. BOHEM and A, H. WAPSTRA: *Phys. Rev.*, **109**, 456 (1958).

RIASSUNTO

Si è misurata la polarizzazione circolare della bremsstrahlung prodotta dai raggi β del ^{32}P con un analizzatore recentemente proposto da BEARD e ROSE. Il grado di polarizzazione trovato è compatibile con le previsioni teoriche. Il funzionamento del polarimetro è risultato soddisfacente.

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On the Analytic Behaviour of the Eigenvalue of the S-Matrix in the Complex Plane of the Energy.

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(ricevuto il 17 Aprile 1958)

Summary. — A study on the analytic behaviour of the eigenvalues of the S-matrix in the complex plane of the energy $E=k^2$ is here carried out for the case of spinless non-relativistic particles scattered by a fixed center of force. A method is developed capable of finding in a finite number of steps any false zero or singularity of the eigenvalues of the S-matrix. The same method shows that in the upper k-plane Jost's $f(k)$ function may have not only simple poles but also any kind of singularity. Our method enables us to establish the main features of these singular points.

1. – Many attempts have been carried out in several papers [1-2] in order to define exactly the relationship between zeros of the eigenvalues of the S matrix and bound states. A zero of $S(k)=f(k)/f(-k)$ in the lower plane of k is necessarily a zero of $f(k)$ (and therefore it corresponds to a bound state) only if $f(-k)$ does not have a pole in the same point. This explains why it is important to know more about the holomorphy domain of $f(-k)$ and therefore of $f(k)$. In the following we shall assume a velocity independent, local potential. For the l-th wave, choosing suitable units, Schrödinger equation can be written as:

$$\psi' + \left|k^2 - \frac{l(l+1)}{x^2} - V(x)\right|\psi = 0\,. \tag{1}$$

(1) R. Jost: *Helv. Phys. Acta*, **20**, 256 (1947).
(2) V. Bargmann: *Rev. Mod. Phys.*, **21**, (2) 488 (1949).

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Following JOST [1] we define an incoming wave as the solution of (1) such that $\lim_{r\to\infty} \exp[-ikx]\,(k, x) = 1$, $k > 0$; $k < 0$ yields outgoing waves. It is also useful to introduce the functions:

$$\lim_{x\to\infty} x^l (2l+1) f(k, x) = f(k) \,. \tag{2}$$

The solution $\varphi(k, x)$ which behaves like x^{l+1} near the origin is given by:

$$\varphi(k, x) = \frac{f(k)\, f(-k, x) - f(-k)\, f(k, x)}{2ik} \,. \tag{3}$$

The eigenvalue of the S matrix is then given by $S(k) = f(k)/f(-k)$.

These function of k defined here on the real axis can be continued in the complex plane P. To shorten the formulas we shall call here $U(\alpha)$ the ensemble of points of P such that $\operatorname{Im}(k) \geqslant \alpha$, $L(\alpha)$ those such that $\operatorname{Im}(k) < \alpha$. The algebra of ensembles will be used throughout the paper. Sum of two ensembles A, B written $A+B$ is the ensemble of points belonging to both. 0 is the empty ensemble.

It is clear that $U(\alpha)+L(\alpha) = P$ and $U(\alpha)\,L(\alpha) = 0$. Instead of « belonging to » we shall write the symbol $\in$.

2. – Some results on the analytic behaviour of $\varphi(k, x)$ and $f(k, x)$ in P wave been derived already rigorously for S waves only [1]. We think it worth to derive them also for any value of l. The first theorem states:

a) $\varphi(k, x)$ is an integer function of k if it possible for any x_0 to find two positive numbers H and ε such that:

$$|V(x)| < \frac{H}{x^{2-\varepsilon}} \,, \qquad x < x_0 \,, \quad \varepsilon > 0 \,. \tag{4}$$

To prove this we have to use the following integral equation, which is equivalent to (1) plus the boundary conditions:

$$\varphi(k, x) = x^{l+1} - \int_0^x \left(\frac{\xi^{l+1}}{x^l} - \frac{x^{l+1}}{\xi^l}\right) [V(\xi) - k^2]\, \frac{\varphi(k, \xi)}{2l+1}\, d\xi \,. \tag{5}$$

The iteration method yields $\varphi(k, x) = \sum_{n=0}^{\infty} \varphi_n(k, x)$, where the $\varphi_n(k, x)$ satisfy the following recurrence relation:

$$\varphi_{n+1}(k, x) = -\int_0^x \left(\frac{\xi^{l+1}}{x^l} - \frac{x^{l+1}}{\xi^l}\right) \frac{V(\xi) - k^2}{2l+1}\, \varphi_n(k, \xi)\, d\xi \,. \tag{6}$$

All these iterated are polinomials in k and therefore analytic, in any finite domain $G \in P$, moreover, if K is the upper limit of the moduli of k^2 in G, one can prove by induction that the $\varphi_n(k, x)$ satisfy the inequalities:

$$|\varphi_n(k, x)| < \left|\frac{2Mx^{\varepsilon}}{\varepsilon(2l+1)}\right|^n \frac{x^{l+1}}{n!}, \qquad M > H + Kx^{2-\varepsilon} . \tag{7}$$

The series $\sum_n \varphi_n(k, x)$ therefore converges uniformly in G and the sum is an analytic function of k in G. Since G is any finite domain $\in P$, $\varphi(k, x)$ is an integer function of k.

b) $f(k, x)$ is analytic in any $G \in L(\alpha)$ and finite on $\text{Im}\,(k) = \alpha$ if one can find for every x_0 a positive number Q such that:

$$|V(k)| < \frac{Q \exp\,[-2\alpha x]}{k^{1+\varepsilon}}, \qquad x > x_0, \quad \varepsilon > 0. \tag{8}$$

To prove this use is made of the following integral equation, which is easily seen to be equivalent to (1) plus the boundary conditions:

$$\begin{aligned} f(k, x) &= f_0(k, x) + \\ &+ \frac{1}{2ik}\int_x^\infty [f_0(-k, x)\, f_0(k, x') - f_0(-k, x') f_0\,(k, x)]\, V(x)\, f(k, x)\, \mathrm{d}x\,. \end{aligned} \tag{9}$$

The solution can be written in the form:

$$f(k, x) = \sum_n f_n(k, x)\,,$$

where

$$f_{n+1}(k, x) = \int_x^\infty G(k, x, x')\, V(x')\, f_n(k, x')\, \mathrm{d}x'\,,$$

$$G(k, x, x') = \frac{1}{2ik}\,[f_0(-k, x)\, f_0(k, x') - f_0(-k, x') f_0(k, x)]\,. \tag{10}$$

Here $f_0(k, x)$ means shortly the solution of the unperturbed equation:

$$\varphi_0'' + \left(k^2 - \frac{l(l+1)}{x^2}\right)\varphi_0 = 0\,, \tag{11}$$

which behaves like $\exp\,[-ikx]$ at ∞. One has $f_0(k, x) = i^{-l-1} H^{(2)}_{l+\frac{1}{2}}(k, x)\sqrt{\pi k x/2}$.

If $\mathrm{Im}\,(k)=\beta$ one can find for any x_0 a number H large enough to have $|f_0(k, x)|< H \exp[\beta x]$ for $x > x_0$. (This can be readily proved by noticing that Hankel's functions are the product of an exponential $\sqrt{2/\pi k x}$ times a polynomial in the inverse power of x). One can then prove by induction that if $k \in L(\alpha)$ $(\beta<\alpha)$ it follows:

$$|f_n| < \frac{H^{2n+1}\exp[\alpha x]}{(\varepsilon\beta x^{\varepsilon})^n n!}\,. \tag{12}$$

$\sum\limits_n f_n(k, x)$ converges uniformly and since the $f_n(k, x)$ (*) are analytical functions of $k \in L(\alpha)$, $f(k, x)$ is an analytic function in any $G \in L(\alpha)$.

3. – Theorems a) and b) are of an extreme value and certainly they are of a great help in deriving many important results of the theory of the S matrix. Yet the second theorem does not yet fully exploit the possibilities inherent to eq. (9). We claim that eq. (9) actually provides a very useful method for the construction of the function $f(k, x)$ also for $k \in U(\alpha)$, singular points being excluded.

Our claim does not rely on any particular use of the solution but rather on the possibility of proving the existence of this solution and of discussing its analytic behaviour in $U(\alpha)$. Actually it is wrong to speak of solution of (9) for $k \in U(\alpha)$ simply because this equation is not satisfied by the function analytic continuation of $f(k, x)$ in $U(\alpha)$ because the integral diverges. What we actually find is a new integral equation whose solution is within certain limits the analytic continuation of eq. (9) out of $L(\alpha)$. To find it we put:

$$f(k, x) = f_0(k, x) + \chi(k, x)\,, \tag{13}$$

from eq. (9) we then deduce:

$$\chi(k, x) = f_1(k, x) + \int\limits_x^\infty G(k, x, \xi)\, V(\xi)\, \chi(k, \xi)\,\mathrm{d}\xi\,. \tag{14}$$

This equation is perfectly equivalent to eq. (9) when $k \in L(\alpha)$. It becomes meaningless when $k \in U(\alpha)$ because the inhomogeneous term $f_1(k, x)$ is represented by a diverging integral (the first Born approximation).

On the other hand it gives a meaningful solution if $f_1(k, x)$ is replaced by a suitable function. The failure of (14) to provide a solution when $k \in U(\alpha)$ is merely due to the lack of definiteness of $f_1(k, x)$. But we can give a meaning to $f_1(k, x)$ for $k \in U(\alpha)$ by defining it as the analytic continuation of $f_1(k, x)$ in $U(\alpha)$. Eq. (14) has now a wider meaning than eq. (9). If now in

(*) The analyticity cannot be proved in $\mathrm{Im}\,(k)=\alpha$, only the finiteness. We did not think it worth reporting here this proof.

some domain G such that $GU(\alpha) \neq 0$ for any given x_0 it is possible to find two numbers M and γ such that $|f_1(k, x)| < M \exp[\gamma x]$, for $x > 0_0$ the same procedure used in deriving theorem (b) shows that (14) defines a solution of (1) which is analytic wherever $f_1(k, x)$ is analytic and provided $k \in L(2\alpha - \gamma)$. The latter condition does not necessarily coincide with $k \in L(\alpha)$ and actually we expect it to be weaker because within $L(\alpha)$ we have $|f_1(k, x)| < M \exp[(\beta - 2\alpha)x]$.

The coefficient γ can be found by direct inspection of the analytic continuation of $f(k, x)$ for $k \in U(\alpha)$. If (as it is like to happen in most practical cases) γ is the same here as in $L(\alpha)$, that is $k = \beta - 2\alpha$ the iterated series converges in $L(2\alpha)$. For simplicity we shall suppose that this will be always the case in the following. We succeded therefore in constructing a solution of (1), analytic in G with the exception of the points in $GU(2\alpha)$ and supposing that no singularity of $f_1(k, x)$ is there contained and that G is simply connected. The latter condition is necessary in order to insure the uniqueness of the analytic continuation of $f_1(k, x)$ in G. As a result the singular points of $\chi(k, x)$ in $L(2\alpha)U(\alpha)$ are all those and only those of $f_1(k, x)$.

Furthermore (14) defines an one valued functional $\chi(k, x)$ of $f_1(k, x)$. It follows that if $f_1(k, x)$ is an one valued function of k the same happens for $\chi(k, x)$. Another obvious conclusion is the following:

Let a function $F(k)$ of k only be given such that $F(k) f_1(k, x)$ is analytic in a domain $G \in L(2\alpha)$. Then $F(k) f(k, x)$ is also regular and analytic in G.

Eq. (14) gives us therefore a tool for the investigation of the analytic properties of the function $f(k)$ and consequently of $S(k)$.

In order to pursue the study of $f(k, x)$ in $U(2\alpha)$ we must use the substitution $f_0(k, x) + f_1(k, x) + \chi_1(k, x) = f(k, x)$ yielding the equation:

$$\chi_1(k, x) = f_2(k, x) + \int_x^\infty G(k, x, \xi)\, V(\xi)\, \chi_1(k, x)\, d\xi \,. \tag{15}$$

The discussion is here quite similar to the one which we have just carried out. This equation enables us to find the singularities of $f(k, x)$ up to $\operatorname{Im}(k) = 3\alpha$ if $|f_2(k, x)| < M \exp[(\beta - 4\alpha)x]$ for $x > x_0$. Using repeatedly this procedure we shall be able to discuss the behaviour of $f(k, x)$ in any finite region of P.

4. – This last part of the paper is devoted to the discussion of $f(k, x)$ for the case of the Yukawa potential $r = B(\exp[-2\alpha x]/\alpha x)$. From the general theorem b) we know at once that $f(k, x)$ will be regular in $L(\alpha)$. Restricting ourselves to S waves eq. (14) writes:

$$\chi(k, x) = f_1(k, x) + \frac{B}{\alpha k} \int_x^\infty \sin k(x - \xi)\, \chi(k, \xi) d\xi \, \frac{\exp[-2\alpha\xi]}{\xi} \,,$$

$f_1(k, x)$ can be evaluated explicitly with the aid of the incomplete γ function or of the integral exponential. The result is:

$$f_1(i\beta, x) = -\frac{B}{2\alpha p}\{\exp[-\beta x]Ein[2(\alpha-\beta)x] -$$
$$-\exp[\beta x]Ein(2\alpha x) - \exp[-\beta x](\ln[2(\alpha-\beta)x) + \exp[\beta x]\ln 2 + 2C ch\beta x\},$$

C is the Mascheroni-Euler constant.

We can split $f_1(k, x)$ into two parts, one is an integer function of k which can be continued everywhere, the other is proportional to $f_0(-k, x)$ and it contains the singularity :

$$f_1(k, x) = I(k, x) + \frac{iB}{\alpha k} f_0(-k, x)\ln\frac{\alpha - ik}{\alpha},$$

$$I(k, x) = -\frac{iB}{2\alpha k}\{\exp[ikx]Ein[2(\alpha + ik)x] - \exp[-ikx]Ein(2\alpha x) +$$
$$-2\sinh ikx\cdot[\ln 2\alpha x + C]\}.$$

If $k \in L(\alpha)$ $f_1(k, x)$ is defined by the integral, for $k \in U(\alpha)$ we define it by analytic continuation. Of course the continuation is carried out in a simple connected domain G non including any singularity of $f_1(k, x)$. From the asymptotic behaviour of $Ein(z)$ we deduce that the iterated series converges in $L(2\alpha)$. The behaviour of $f(k, x)$ around the singularity $k = i\alpha$ can be best seen by splitting $\chi(k, x)$ into two parts:

$$\chi(k, x) = \lambda(k, x) + \frac{iB}{k\alpha}\ln\frac{\alpha + ik}{\alpha}\mu(k, x),$$

obeying the equations:

$$\lambda(k, x) = I(k, x) - \frac{B}{i\alpha k}\int_x^\infty \sin k(x - x')\lambda(k, x')\frac{\exp[-2\alpha k']}{x'}dx',$$

$$\mu(k, x) = f_0(-k, x) + \int_x^\infty G(k, x, x')V(x')\mu(k, x')\,dx',$$

$\lambda(k, x)$ is analytic in $L(2\alpha)U(-\alpha')$ including $k = i\alpha$, with $\alpha' < \alpha$, $\mu(k, x)$ is clearly $f(-k, x)$ which is analytic in $U(-\alpha')$. We can write for $k \in L(2\alpha)U(-\alpha')$:

$$\chi(k, x) = \lambda(k, x) - \frac{B}{i\alpha k}\ln\frac{\alpha + ik}{\alpha}f(-k, x).$$

In $L(-\alpha')$ both $\lambda(k, x)$ and $f(-k, x)$ have singularities and this is the reason why their iterated series do not converge. These singularities cancel each other in the expression for $\chi(k, x)$ so that the latter is regular in $L(-\alpha')$.

The point $k = i\alpha$ is then a logarithmic singularity of $f(k, x)$ as expected. This is a rigorous proof of Jost's conjecture (1). Further iterations of eq. (9) are needed to study $f(k, x)$ in $U(2\alpha)$ but the integrals to be calculated are too complicated to be examined here.

* * *

I thank Prof. M. VERDE for his kind interest in this work and for many advices.

APPENDIX I.

We give here some formulas regarding the definition and the behaviour of the function $Ein(z)$:

$$Ein(z) = \int_0^z \frac{1 - \exp[-t]}{t}\, dt = z \sum_0^\infty \frac{(-)^n z^n}{n n!} .$$

$Ein(z)$ is therefore an integer function of z.

$$Ein(z) = \ln z + C - Ei(z)\,, \qquad C = \lim_{m\to\infty}\left(\sum_1^m \frac{1}{q} - \ln m\right) \simeq 0.5772...\,,$$

$$Ein(z) \sim \ln z + C + \frac{\exp[-z]}{z} \sum_n \frac{(-)^n n!}{z^n} \qquad \text{for} \quad |z| \to \infty.$$

APPENDIX II.

We show here how to derive (7) and (12). We firstly suppose that for a given n one has:

$$|\varphi_n(k, x)| < Q_n x^{1+n\varepsilon-l}$$

and then we try to derive it for $\varphi_{n+1}(k, x)$. Obviously:

$$\left|\frac{x^{l+1}}{\xi^l} - \frac{\xi^{l+1}}{x^l}\right| < 2\,\frac{x^{l+1}}{\xi^l}\,, \qquad x > \xi.$$

Also $|V(\xi)-k^2|< H\xi^{\varepsilon-2}+K< M\xi^{\varepsilon-2}$,

$$|_{n+1}(k,x)|<\int_0^x d\xi\cdot Q_n\xi^{1+n\varepsilon+l}|V(\xi)-k^2|\frac{1}{2l+1}2\frac{x^{l+1}}{\xi^l}<$$

$$<2\int_0^x d\xi\cdot Q_n M\xi^{1+(n+1)\varepsilon-2}\frac{1}{2l+1}x^{l+1}<\frac{2Q_nM}{\varepsilon(n+1)(2l+1)}x^{\varepsilon(n+1)+1+l}.$$

We finally get the recurrence relation with $Q_0=1$:

$$Q_{n+1}=\frac{2Q_nM}{\varepsilon(2l+1)(n+1)},$$

whose solution is $Q=(2M/\varepsilon(2l+1))^n(1/n!)$ as stated.

We proceed quite similarly in deriving (12). We assume that for a given n the inequality holds:

$$|f_n(k,x)|< R_n\exp[\alpha x]x^{\varepsilon-n}, \qquad k=i\beta+\gamma.$$

Clearly:

$$|f_{n+1}(k,x)|< R_n\int_x^\infty \exp[\alpha\xi]\xi^{-\varepsilon n}d\xi|V(\xi)|G(k,x,\xi)<$$

$$<\frac{R_nH^2}{(k)}\int_x^\infty d\xi\cdot Q\frac{\exp[-2\alpha\xi]}{\xi^{1+\varepsilon}}\exp[\beta(\xi-x)]<\frac{R_nH^2Qx^{-\varepsilon(n+1)}\exp[\alpha x]}{|k|\varepsilon(n+1)},$$

so that

$$|f_n|< H\exp[\alpha x]\left[\frac{H^2Q}{|k|\varepsilon x^\varepsilon}\right]^n\frac{1}{n!}.$$

RIASSUNTO

Si propone un metodo per lo studio del comportamento della funzione di Jost nel piano dell'energia complessa allo scopo di mettere in evidenza il tipo di singolarità che detta funzione possiede a seconda del potenziale scelto. Tale conoscenza è importante per la costruzione di un criterio che permetta di eliminare senza ambiguità quegli zeri della matrice S che non corrispondono a stati legati. Il metodo proposto è più applicato al caso di onde S con il potenziale di Yukawa.

Measurement of the Wind Produced in Liquids by an Excited Piezoelectric Quartz.

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(ricevuto il 2 Maggio 1958)

Summary. — In this note are described the measurements made on the water for ultrasonic frequencies of 5.5 and 6.6 MHz in order to determine the wind produced by the ultrasonic current. The measurements of the velocity acquired by the fluid is obtained through measurements of the intensity of the ultrasonic beam. The results show the presence, also for these frequencies, of two fluid flows and moreover allow the determination of the ratio of the two viscosity coefficients. These values do not correspond to those obtained from the determination of the absorption coefficient of the ultrasounds, therefore it is necessary to make other measurements for higher frequencies.

In a previous note (1) we described an experimental device by means of which starting from Eckart's formula, that gives the velocity v acquired by a fluid enclosed within a cylinder of radius r_0 and along the axis of which an ultrasonic beam runs having an intensity I, and radius r it is possible to measure the value η'/η that is considered equal to the ratio between the two viscosity coefficients (respectively related to the bulk viscosity and to the sheer viscosity); the formula is the following:

$$v = G\,\frac{r^2\omega^2 I}{\varrho c^4}\left(2+\frac{\eta'}{\eta}\right), \tag{1}$$

in which: $G=\frac{1}{2}[(r/r_0)^2-1]-G(r/r_0)$, ϱ is the density, c the velocity of the sound in the liquid under examination and ω the pulsation of the wave.

(1) A. Carrelli and F. Cennamo: *Nuovo Cimento*, **1**, 365 (1955).

In the device used in the present research and in the preceding one, in contrast with the devices used for previous researches ([2]), the measures of I and of v are always taken at the same time, but independently. The first is obtained by measuring the radiating pressure produced by the ultrasonic beam that comes out of the cylinder through a screen allowing ultrasonics to pass, on a disc at an angle of 45° with the direction of the beam and the reflecting power R of which has been directly measured. The velocity acquired through the effect of the ultrasonics is determined by the aspect taken under the influence of the ultrasonic beam by a stream line of coloured water flowing at a known velocity along the vertical diameter of the tube and, therefore, at right angles with the direction of the ultrasonic wind.

This method that calls also for the experimental determination of the reflecting power R of the disc, gives a more reliable value of I' and, moreover, ensures that the two essential premises of Eckart's theory are applied, namely: perfect sealing of the tube in which the ultrasonics are produced, and the practically null reflecting power of the wall limiting the tube on the opposite side to that on which the quartz is fixed.

Moreover the present device affords also the possibility of deriving from the shape acquired by the coloured stream, the distribution of the speed v of the liquid, in consequence of the ultrasonics, along a diameter of the tube.

In view of the fact that during previous researches ([2]), while measuring the value of η'/η of various liquids and at various frequencies it became apparent that the said value varied with the varying of the frequency, and, again, in view of the fact that the meaning of η' is not yet theoretically very clear, we decided to proceed to further measurements of the ratio η'/η, at least in relation to water, at frequencies of $\nu = 5.4$ MHz and $\nu = 6.6$ MHz and in relation with different values of the intensity I of the ultrasonic beam.

It must be noted however that while the measurement of v takes place inside the tube, the measurement of the intensity is made on the outside, *i.e.* after the ultrasonic beam generated by the quartz has travelled both a notable thickness of liquid and the cellophane lamina sealing off the tube.

The intensity I, determined by the measurement of the radiation pressure generated by the ultrasonic beam on the reflecting lamina, must therefore be corrected both because of the absorption caused by the thickness x of the liquid crossed, and this is obtained by multiplying the measured value I by $\exp[2\alpha x]$ (α is the absorption coefficient), and because of the transparency of the cellophane lamina.

For the correction due to the absorption of the water the values selected for α were those obtained through the thermic method by Grossetti ([3]) that

([2]) A. Carrelli and F. Cennamo: *Nuovo Cimento*, **11**, 429 (1954); **12**, 1 (1954).
([3]) E. Grossetti: *Nuovo Cimento*, **11**, 250 (1954).

carried out his measurements in the same field of frequencies in which we worked.

In order to determine, instead, the transparency factor of the cellophane lamina sealing the end of the tube and hit by the ultrasonic at a normal angle, we made use of the known formula:

$$R = \frac{\left(\frac{\varrho_1 c_1}{\varrho_2 c_2} - \frac{\varrho_2 c_2}{\varrho_1 c_1}\right)^2}{4 \operatorname{cotg}^2 \frac{2\pi s}{\lambda} + \left(\frac{\varrho_1 c_1}{\varrho_2 c_2} + \frac{\varrho_2 c_2}{\varrho_1 c_1}\right)^2},$$

where: $\varrho_1 c_1$ and $\varrho_2 c_2$ are respectively the acoustic impedances of the water and the cellophane, s the thickness of the cellophane lamina, λ is the wavelength of the ultrasonics in the cellophane. The lamina of cellophane that was used had a thickness of $s = 3 \cdot 10^{-3}$ cm. The acoustic impedance of the cellophane was calculated by measuring the densities ϱ_2 of the sheet of cellophane from which the lamina under consideration had been taken; the speed c_2 of the ultrasonics in the cellophane was obtained through the formula:

$$c_2 = \sqrt{\frac{k + \frac{4}{3}\eta}{\varrho}},$$

where k and η are respectively the compressibility and the rigidity coefficients of the cellophane, and these coefficients were determined by means of Young's measurements of the coefficient E with the commonly used method in traction elasticity and by means of the method of torque, the latter determined by means of the method of torque oscillations from the relation:

$$M = \mu \alpha,$$

where μ is the unitarian torque moment in a cellophane lamina having a length of l and a rectangular transverse section with sides a, b, given by:

$$\mu = \frac{1}{3} \frac{ab^3}{l} n,$$

Fig. 1 contains some curves regarding the distribution of the intensities obtained for the frequency $\nu = 5.4$ MHz at various intensities of the ultrasonic beam; the state of the distribution of the speeds in relation to the central part of the tube, which was the zone where the measurements could be effected (see Fig. 4, of Note III), corresponds generally speaking to the theoretical forecasts.

Fig. 2 contains the values of the velocity v, measured in correspondance to the axis of the tube in relation with the I of the two frequencies studied.

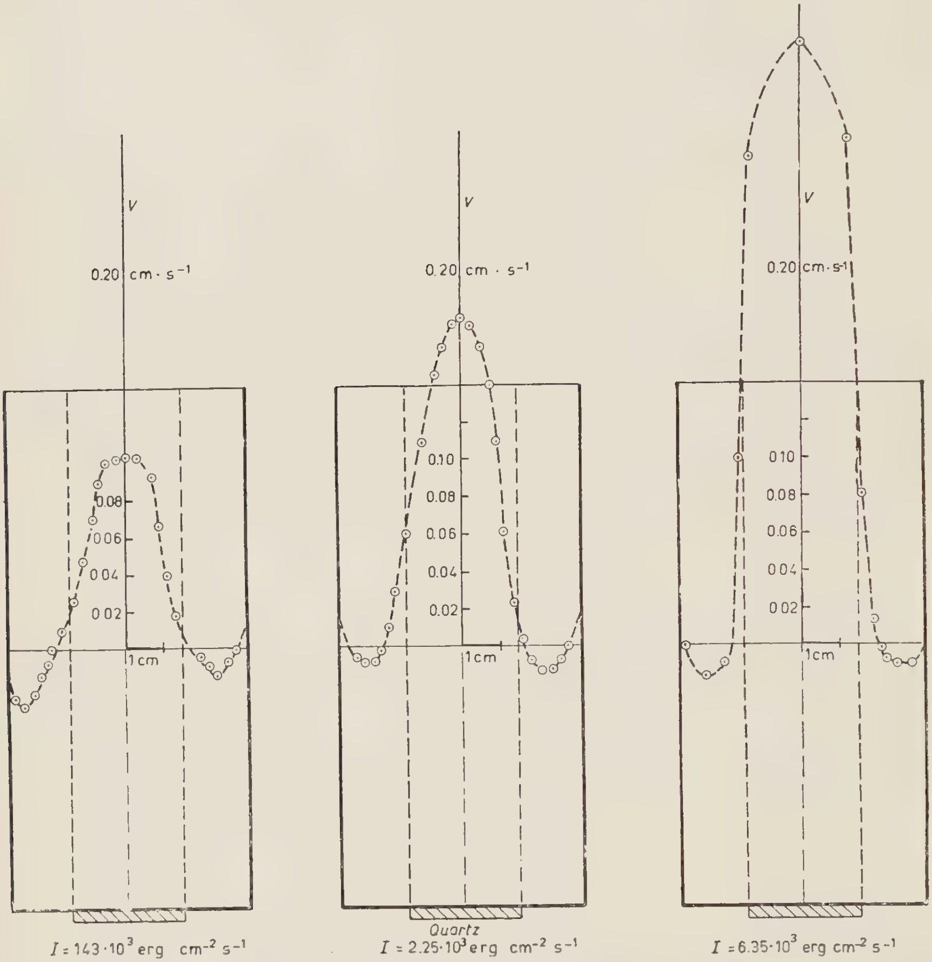

Fig. 1.

For the frequency $\nu = 6.6$ MHz the representative points arrange themselves along two straight lines having two distinct inclinations one for the low and the other for the high sound intensities; this result, obtained by means of the device in question which is evidently much more exact, goes to confirm what had been previously observed, and stressed as well by LIEBERMAN, it proves, that is, the existence of a distinct regime of motion for greater intensities; on the other hand we can state, in view of the results that we achieved, that these intensities, at which another regime of motion appears, are all the

lower the higher the frequency. From the determination of the angular coefficients of the straight lines related to the low frequencies of the ultrasonic beam, taking into account the corrections that must be made to I, it is possible, on the ground of 1), to go back to the value of η'/η.

The values experimentally obtained for η'/η are contained in the third column of the following table; from it there appears a dependence of η'/η on the frequency in the sense that the ratio in question increases as the frequency increases. This is in complete accord with the results that we had obtained during the course of previous researches.

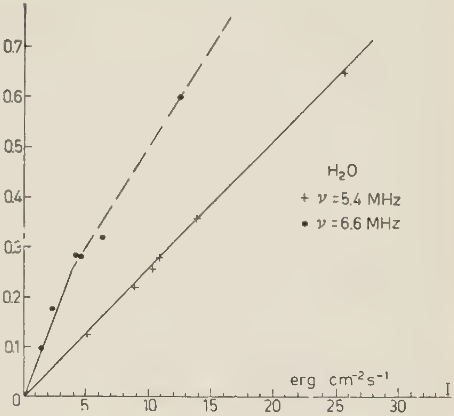

Fig. 2.

r	α/γ^2	η'/η exper.	η'/η calc.	Remarks
$1.8\cdot10^6$ Hz	$87.0\cdot10^{-17}$ cm^{-1}	1.40	12.0	2nd and 3rd note
3.0 »	47.0 »	2.30	7.6	2nd note
4.2 »	31.0 »	3.80	4.3	2nd »
5.4 »	24.5 »	4.32	3.0	4th »
6.6 »	21.0 »	6.97	2.3	4th »

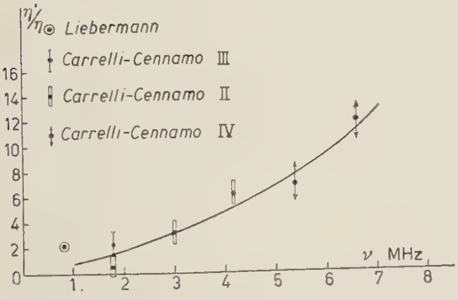

Fig. 3.

It is to be noted that the absorption coefficient α is given by:

$$\alpha = \frac{\omega^2\eta}{2\varrho c^3}\left(2 + \frac{\eta}{\eta'}\right),$$

in which: $\omega = 2\pi\nu$ is the pulsation of the wave, ϱ and c respectively the density of the liquid and the velocity of the ultrasonics; it is therefore possible to obtain the value of the ratio η'/η by a completely different method, based, that is, on the experimental values of ϱ and c, and of the absorption coefficient α.

We must notice that we utilize the data obtained by GROSSETTI as the measures are all made by the same method. On the other side if we utilize data

obtained by other authors, we will achieve results which give us some conclusions still more marked than those we attained. The ratio η'/η calculated on the basis of such data for the various frequencies is indicated in the column second from the last.

In comparing the data η'/η obtained from the absorption coefficient to those obtained with the measures we present, it is possible to notice that by measuring the absorption coefficient we obtain data of η'/η which reduce as the frequency increases; from the measurements carried out by means of the method of the quartz-wind the opposite is instead achieved (Fig. 3) at least for what concerns water and in this frequency field.

Researches will be made with other liquids in order to establish their behaviour regarding the variation of the frequency.

RIASSUNTO

In questa nota sono riportate le misure compiute sull'acqua per frequenze ultrasonore di 5.4 e 6.6 MHz, per la determinazione del vento prodotto dalla corrente ultrasonora. La misura della velocità acquistata dal fluido è ottenuta con misure dirette dell'intensità del fascio ultrasonoro. I risultati mostrano l'esistenza, anche per queste frequenze, di due regimi di moto, e inoltre permettono una determinazione del rapporto dei due coefficienti di viscosità. Questi valori non corrispondono a quelli ricavati dalla determinazione del coefficiente di assorbimento degli ultrasuoni per tale frequenza: è necessario quindi fare altre misure per frequenze più elevate.

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Babinet's Principle for Diffraction at a Plane Screen with Directional Conductivity (*).

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(ricevuto il 5 Maggio 1958)

Summary. — A plane and infinite thin screen Σ_1 of any type of connection is a part of a plane $\Sigma_1+\Sigma_2$. A system of regular lines (lines of conductivity) is defined over Σ_1, such that through each point of Σ_1 there passes one line of the system, and only one. Conductivity is assumed to be infinite along the lines of conductivity, and to vanish along their normals. A general electromagnetic wave is incident upon Σ_1. A complementary case, in the sense of Babinet's principle, is obtained when Σ_1 has still the same properties, while Σ_2 becomes a screen of infinite and omnidirectional conductivity.

1. – Introduction.

A plane and infinitely thin screen will be said to have *directional* conductivity, when at each one of its points a direction is defined in which conductivity is infinite (both ways), while conductivity perpendicular to that direction is zero. We will confine ourselves to the case when all the directions of conductivity are tangent to a set of regular and non-intersecting lines, which will be termed the *lines of conductivity*. A screen with infinite conductivity in all directions will be said to have *ordinary conductivity*.

The particular case when the lines of conductivity are parallel and straight lines (*uni-directional conductivity*) has recently been the subject of some in-

(*) This research has been sponsored by the Air Research and Development Command, United States Air Force, under Contract AF 61(052)-67, through the European Office A.R.D.C.

vestigations. The author has discussed diffraction by a circular disc of small radius (1) and given the analogous of Schwinger's variational principle for problems of this kind (2). KARP (3) has obtained an exact solution for the diffraction of a plane wave by a semi-infinite screen, while RADLOW (4) has solved the same problem for an incident dipole field. The author has also discussed the analogous of Babinet's principle for the case of unidirectional conductivity (5).

The purpose of the present paper is to extend Babinet's principle to the case when the lines of conductivity have a general shape.

2. – Conditions for diffraction by a directional screen.

Let the directional screen Σ_1 (Fig. 1) occupy a portion of the plane $\Sigma = \Sigma_1 + \Sigma_2$, bounded by a regular contour Γ, all at finite distance. It is not necessary to require that Σ_1 should be simply connected.

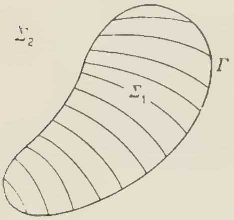

Fig. 1.

We shall refer to a set of right-handed, rectangular, curvilinear coordinates u_1, u_2, u_3, defined as follows: $u_3 = z$ is simply equal to the distance from the plane Σ, while u_1, u_2 are restricted only by the requirement that on Σ_1 the lines $u_2 = constant$ should coincide with the lines of conductivity. There are infinitely many sets of coordinates which satisfy this condition. A line element ds will be assumed to have the expression $\mathrm{d}s^2 = h_1^2 \mathrm{d}u_1^2 + h_2^2 \mathrm{d}u_2^2 + \mathrm{d}z^2$, where h_1, h_2 are, of course, functions of u_1 and u_2.

Let the incident field be represented by $\boldsymbol{E}^i$, $\boldsymbol{H}^i$. The time dependence $\exp[-i\omega t]$ will be understood. The total field will be represented by

$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s\,, \qquad \boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s\,, \tag{1}$$

where $\boldsymbol{E}^s$, $\boldsymbol{H}^s$ is the scattered field.

The boundary conditions to be applied are those which were « guessed » by the author in his earlier works and were later confirmed by the exact analyses

(1) G. TORALDO DI FRANCIA: *Nuovo Cimento*, **3**, 1276 (1956); **6**, 150 (1957).
(2) G. TORALDO DI FRANCIA: *Rend. Acc. Naz. Linc.*, **21**, 86 (1956).
(3) S. N. KARP: *New York Univ., Inst. Math. Sc., Electrom. Div., Rep. No. EM*-108.
(4) J. RADLOW: *New York Univ., Inst. Math. Sc., Electrom. Div., Rep. No. EM*-105.
(5) G. TORALDO DI FRANCIA: *Rend. Acc. Naz. Linc.*, **20**, 476 (1956).

of KARP (3) and RADLOW (4):

(2) $$E_1^s = -E_1^i \quad \text{on } \Sigma_1, \qquad E_1^s \quad \text{continuous across } \Sigma_2 .$$

(3) $$E_2^s \quad \text{continuous across } \Sigma .$$

(4) $$H_1^s = 0 \quad \text{on } \Sigma .$$

(5) $$H_2^s = 0 \quad \text{on } \Sigma_2, \qquad H_2^s(z = +0) = -H_2^s(z = -0) \text{ on } \Sigma_1 .$$

(6) $H_2^s \to 0$ on Σ_1 at the edge Γ (except where Γ coincides with a line of conductivity),

Physically, (2) is the condition of directional conductivity, (3) expresses the absence of magnetic currents, (4) and (5) express the fact that the magnetic field generated by any element of current induced over Σ_1 is perpendicular to Σ except right at the element, where it has the direction of u_2 and equal and opposite values on both sides of Σ_1, and (6) requires that the component of the current perpendicular to Γ should vanish. As a consequence of (6), there is no accumulation of charge on the edge. To obtain a unique solution, we will further require that H_2^s should vanish as the square root of the distance from the edge. The condition at infinity will be the radiation condition which requires that the scattered field should behave like a divergent wave.

The current induced over Σ_1 has the surface density $-2H_2^s(u_1, u_2 + 0) = 2H_2^s(u_1, u_2, -0)$ and is obviously in the direction of u_1. Therefore, the vector potential has the expression

(7) $$\boldsymbol{A}(u_1, u_2, z) = -2 \iint_{\Sigma_1} H_2^s(u_1', u_2', +0)\, G(r)\, \boldsymbol{i}_1' h_1' h_2' \, \mathrm{d}u_1' \, \mathrm{d}u_2' ,$$

where $\boldsymbol{i}_1'$ is a unit vector in the direction of u_1, taken at the point of integration $(u_1', u_2', 0)$, r is the distance between the point of integration and the point (u_1, u_2, z), and the free-space Green function $G(r)$ is defined by

(8) $$G(r) = \frac{\exp[ikr]}{4\pi r} ,$$

with $k = 2\pi/\lambda = \omega/c$.

By applying standard formulae and noting that $\boldsymbol{A}$ is parallel to Σ ($A_3 = 0$),

we shall obtain the scattered field in the form

$$E_1^s = -\frac{Z}{ikh_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1h_2}\left[\frac{\partial h_2A_1}{\partial u_1}+\frac{\partial h_1A_2}{\partial u_2}\right]\right\}+ikZA_1\,, \tag{9}$$

$$E_2^s = -\frac{Z}{ikh_2}\frac{\partial}{\partial u_2}\left\{\frac{1}{h_1h_2}\left[\frac{\partial h_2A_1}{\partial u_1}+\frac{\partial h_1A_2}{\partial u_2}\right]\right\}+ikZA_2\,, \tag{10}$$

$$E_3^s = -\frac{Z}{ik}\frac{1}{h_1h_2}\frac{\partial}{\partial z}\left[\frac{\partial h_2A_1}{\partial u_1}+\frac{\partial h_1A_2}{\partial u_2}\right], \tag{11}$$

$$H_1^s = -\frac{\partial A_2}{\partial z}\,, \tag{12}$$

$$H_2^s = \frac{\partial A_1}{\partial z}\,, \tag{13}$$

$$H_3^s = \frac{1}{h_1h_2}\left[\frac{\partial h_2A_2}{\partial u_1}-\frac{\partial h_1A_1}{\partial u_2}\right], \tag{14}$$

where Z denotes the impedance of free space.

The radiation condition is satisfied because of (8). Condition (2) in conjunction with (9) requires the following equation to be satisfied at all points $(u_1, u_2, 0)$ of Σ_1

$$\frac{1}{h_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1h_2}\left[\frac{\partial h_2A_1}{\partial u_1}+\frac{\partial h_1A_2}{\partial u_2}\right]\right\}+k^2A_1=\frac{ik}{Z}E_1^i\,. \tag{15}$$

By recalling (7), it is seen that (15) represents an integro-differential equation for the function $H_2^s(u_1, u_2, +0)$. Conditions (3), (4), (5) are automatically satisfied [6]. Thus the problem is reduced to finding a solution of (7) and (15) which satisfies the condition (6).

3. – The complementary problem.

We now want to discuss the case when the directional screen Σ_1 considered in the foregoing section is surrounded by a screen with *ordinary* conductivity Σ_2, which occupies the remaining portion of Σ (Fig. 2).

[6] See, for instance: G. TORALDO DI FRANCIA: *Introduction to the Modern Theory of Electromagnetic Diffraction* (Pubbl. Ist. Naz. Ottica, Firenze, 1956), p. 28.

Let the field $\boldsymbol{E}^i$, $\boldsymbol{H}^i$ be incident on Σ, from the side $z<0$. We will write the total field in the form

$$\boldsymbol{E}=\boldsymbol{E}^i+\boldsymbol{E}^r+\boldsymbol{E}^s\,,\qquad \boldsymbol{H}=\boldsymbol{H}^i+\boldsymbol{H}^r+\boldsymbol{H}^s\,, \tag{16}$$

where $\boldsymbol{E}^r$, $\boldsymbol{H}^r$ represents the field which would be reflected by a whole plane Σ with ordinary conductivity, and $\boldsymbol{E}^s$, $\boldsymbol{H}^s$ is an additional field which (conventionally) will be called the scattered field.

By recalling that the tangential part of $\boldsymbol{E}^i+\boldsymbol{E}^r$ vanishes at Σ, while the tangential part of $\boldsymbol{H}^i+\boldsymbol{H}^r$ equals twice that of $\boldsymbol{H}^i$, it is seen that all the necessary conditions of continuity across Σ are satisfied by requiring

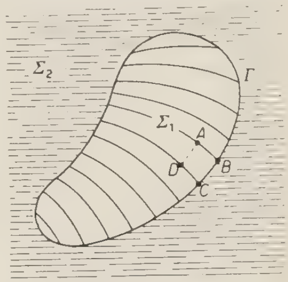

Fig. 2.

$$E_1^s=0 \quad \text{on} \quad \Sigma\,, \tag{17}$$

$$E_2^s \ \text{continuous across}\ \Sigma_1\,,\qquad E_2^s=0 \quad \text{on}\ \Sigma_2\,, \tag{18}$$

$$\begin{cases} H_1^s(z=+0)=H_1^i\,, \qquad H_1^s(z=-0)=-H_1^i & \text{on}\ \Sigma_1\,,\\ H_1^s(z=+0)=-H_1^s(z=-0) & \text{on}\ \Sigma_2\,, \end{cases} \tag{19}$$

$$H_2^s(z=+0)=-H_2^s(z=-0) \qquad \text{on}\ \Sigma\,, \tag{20}$$

$$E_2^s\to 0 \quad \text{on}\ \Sigma_1 \ \text{at the edge}\ \Gamma \ \text{(except where}\ \Gamma\ \text{coincides with a line of conductivity).} \tag{21}$$

Not all of these conditions are necessary *a priori*, however we have written them down for completeness.

It is readily verified that all the above conditions, except (19), can be met by assuming that the scattered field in the half space $z>0$ is generated by a fictitious magnetic current on Σ_1 having the surface density $2E_2^s(u_1, u_2, 0)$ and the direction of u_1, while the scattered field in the half space $z<0$ is generated by an equal and opposite magnetic current. Condition (21) requires that no magnetic charges should be accumulated at Γ (for the physical meaning of this condition see the conclusion). To obtain a unique solution we shall require that, when approaching Γ the magnetic current should behave like the square root of the distance from Γ.

In the half space $z>0$ we shall have a magnetic vector potential $\boldsymbol{B}$ given by

$$\boldsymbol{B}(u_1, u_2, z)=2\iint\limits_{\Sigma_1} E_2^s(u_1, u_2, 0)\, G(r)\, \boldsymbol{i}_1' h_1' h_2'\, \mathrm{d}u_1'\, \mathrm{d}u_2'\,. \tag{22}$$

From this, the field components are obtained

$$E_1^s = \frac{\partial B_2}{\partial z}\,, \tag{23}$$

$$E_2^s = -\frac{\partial B_1}{\partial z}\,, \tag{24}$$

$$E_3^s = -\frac{1}{h_1 h_2}\left[\frac{\partial h_2 B_2}{\partial u_1}\;\frac{\partial h_1 B_1}{\partial u_2}\right], \tag{25}$$

$$ZH_1^s = -\frac{1}{ikh_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1 h_2}\left[\frac{\partial h_2 B_1}{\partial u_1} + \frac{\partial h_1 B_2}{\partial u_2}\right]\right\} + ikB_1\,, \tag{26}$$

$$ZH_2^s = -\frac{1}{ikh_2}\frac{\partial}{\partial u_2}\left\{\frac{1}{h_1 h_2}\left[\frac{\partial h_2 B_1}{\partial u_1} + \frac{\partial h_1 B_2}{\partial u_2}\right]\right\} + ikB_2\,, \tag{27}$$

$$ZH_3^s = -\frac{1}{ikh_1 h_2}\frac{\partial}{\partial z}\left[\frac{\partial h_2 B_1}{\partial u_1} + \frac{\partial h_1 B_2}{\partial u_2}\right]. \tag{28}$$

Condition (19) requires the following equation to be satisfied at all points $(u_1, u_2, 0)$ of Σ_1

$$\frac{1}{h_1}\frac{\partial}{\partial u_1}\left\{\frac{1}{h_1 h_2}\left[\frac{\partial h_2 B_1}{\partial u_1} + \frac{\partial h_1 B_2}{\partial u_2}\right]\right\} + k^2 B_1 = -\,ikZH_1^i\,, \tag{29}$$

which, in conjunction with (22), represents an integro-differential equation for $E_2^s(u_1, u_2, 0)$.

It is now an easy matter to show that all the equations of the present section, written for $z > 0$, transform into those of the previous section if the substitution is made

$$\boldsymbol{E}^i \to \;\; Z\boldsymbol{H}^i\,, \qquad Z\boldsymbol{H}^i \to -\boldsymbol{E}^i\,, \tag{30}$$

$$\boldsymbol{E}^s \to -Z\boldsymbol{H}^s\,, \qquad Z\boldsymbol{H}^s \to \;\; \boldsymbol{E}^s\,. \tag{31}$$

These equations show that the screens of Fig. 1 and 2 are complementary in the sense of Babinet's principle, as corrected by COPSON [7] and MEIXNER [8].

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(7) E. T. COPSON: *Proc. Roy. Soc.*, **186**, 100 (1946).
(8) J. MEIXNER: *Ann. der Phys.*, **6**, 2 (1949).

4. – Conclusion.

We have shown that the directional screen Σ_1 of Fig. 1 is complementary in the sense of Babinet to the screen of Fig. 2, which is constituted by the directional screen Σ_1 plus the screen with ordinary conductivity Σ_2. Both the integro-differential equation and the boundary conditions of one case transform into those of the other by means of the substitutions (30), (31).

The physical meaning of condition (6) is obvious. It is of interest to show the physical meaning of the corresponding condition (21). Consider a contour like $ABCD$ of Fig. 2. The electromotive force acting upon this contour is simply equal to the integral of E_2^s along AD; it is also equal in absolute value to the magnetic displacement current across the area $ABCD$. Therefore condition (21) states that the integral of H_z over the area $ABCD$ must vanish when AD approaches BC. In other words, H_z may become infinite at the edge Γ, provided it is integrable, a familiar condition in diffraction problems. Incidentally, we note that this proves that Σ_1 and Σ_2 must be in contact along Γ.

RIASSUNTO

Uno schermo piano infinitamente sottile Σ_1, con qualsiasi tipo di connessione, occupi una parte del piano $\Sigma_1 + \Sigma_2$. Su Σ_1 sia definito un sistema di linee regolari (linee di conduttività), in modo che per ciascun punto di Σ_1 passi una e una sola linea del sistema. Si supponga che la conduttività di Σ_1 sia infinita lungo le linee di conduttività e nulla in direzione perpendicolare ad esse, un'onda elettromagnetica di forma qualsiasi incida su Σ_1. Si ottiene il caso complementare nel senso di Babinet quando Σ_1 ha le stesse proprietà descritte, mentre il resto del piano Σ_2 diviene uno schermo con conduttività ordinaria, onnidirezionale.

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On the Charge Conjugation of Baryons (*).

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(ricevuto il 17 Maggio 1958)

Summary. — If we consider the baryon as composed by a bare spinor without charge nor strangeness and the known meson fields (both π and K) one may investigate the behaviour of the lagrangian describing this model under an operation which reverses the sign of charge and strangeness of the mesons wihout altering the bare baryon (boson conjugation B) and an operation which turns the bare baryon into its antiparticle without changing the meson fields (spinor conjugation S), the product of these two operations being the usual charge conjugation C. It is found, when we neglect mass differences between nucleon and hyperons, that the lagrangian of this system is invariant for each of the three operations B, S and C separately. The same property is deduced also for the d'Espagnat-Prentki lagrangian of an assembly of real nucleons, hyperons and mesons and for the interactions with the electromagnetic field of all these particles. Finally it is suggested that the mass difference between nucleon and Ξ could be due to an interaction term in the lagrangian which should be invariant under charge conjugation but not under the B and S conjugations separately and a simple proposal for such a term is made.

1. – It is well known that in pure electrodynamics, the operation leading to charge conjugation is achieved by linking the transformation which reverses the sign of the electromagnetic field with the transformation which converts the particles into antiparticles [1].

(*) This work in a preliminary form has been presented at the *Padua-Venice Conference on Mesons and recently discovered Particles*, September 1957.

[1] A. Pais and R. Jost: *Phys. Rev.*, **87**, 871 (1952); L. Wolfenstein and D. G. Ravenhall: *Phys. Rev.*, **88**, 279 (1952).

Up to now, the same formalism has been assumed to hold also for the charge conjugation of nucleons, and by implicit extension of all kind of baryons; that is, it has been assumed that also for baryons, the reversal of the sign of the electric charge implies the simultaneous transformation of the baryon into the corresponding antibaryon. As the transformation of a baryon into an antibaryon should imply the change of sign of the baryon number, this means that in the operation of charge conjugation, it is implicitly assumed that electric charge and baryon number are linked together in order that they both simultaneously reverse their sign when charge conjugation is applied.

This however, does not seem to be a necessary condition, at least for some possible baryon models; moreover, a number of different baryon states has been in these last years experimentally observed, and if, according to the suggestion of different authors, we disregard their mass differences, these different states may be tried to be accounted for as a consequence of general symmetry properties applied to the baryons. In this line of thought one could therefore be tempted to consider what kind of general properties or symmetries should be expected and obtained when we define operations in which either the sign of the electric charges is reversed without changing the baryon number or the baryon number is reversed without changing the electric charges; and whether some known experimental baryon state would fit into these schemes. In order to develop first these considerations on a well defined case, we will try in the present work to define and apply these possible new operations on nucleons. It will be found that the result of the first operation (change of charge without changing baryon number) which will be called the boson conjugation B transforms, when differences of mass are disregarded, the nucleons into the Ξ's, while the result of the second operation (change of baryon number without changing electric charges) which will be called the spinor conjugation S transforms the nucleons into anti Ξ's, while of course these two combined operations, which are equivalent to usual particle-antiparticle conjugation C, transform the nucleons into antinucleons and Ξ's into anti Ξ's. The results so achieved will be extended to all baryons, and finally the effect of possible reasons for the original mass difference will be briefly discussed.

2. – As a first point, we must stress our that our aim is not to look for just a mathematical possibility, allowing to write formally the transformations corresponding to our preceding definitions but to point out what their physical meaning could be as related to what we may call the structure of the nucleon. In this sense, the easiest way to obtain some physical insight in the problem we are looking for, may consist in assuming some model for the nucleon, adequate to underline the fundamental features of its constitution, in agreement with our present knowledge. We shall adapt to this aim a model that has al-

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ready been proposed [2], consisting of a bare spinor baryon B_0 with no isospin and no strangeness, which is the source of both the K and the pion field; the real baryons are then obtained by clothing the bare B_0 both with K's and π's. This gives the possibility of constructing all the different states of electric charge and strangeness that have been observed up to now. The nucleons in particular will then be obtained by clothing first B_0 with the two K-mesons of positive strangeness, K^+ and K_0 obtaining thus two states N_0 (N_0^+ and N_0^0) and then letting pions being exchanged between these states in order to obtain the real proton and neutron. In symbols we will have:

$$
(1)\qquad
\begin{aligned}
&N_0 = B_0 + K\,, && N_0^+ = B_0 + K^+\,, && N_0^0 = B_0 + K^0\\
&N \;= N_0 + \pi\,, && P \;= N_0^0 + \pi^+ = N_0^+ + \pi^0\,, && N \;= N_0^+ + \pi^- = N_0^0 + \pi^0\,.
\end{aligned}
$$

It may be observed that by strangeness we mean here what has been termed by SCHWINGER [3] « hypercharge » and differs by one positive unit from Gell-Mann's strangeness.

Let us first remark that the nature of the K-meson field is such that for mesons positive charge is always associated with positive strangeness, and negative charge with negative strangeness, and never the reverse (*i.e.* only K^+ and K^- exist). Thus a reversal of the charge for all the mesons can only happen if also there is a simultaneous reversal of the strangeness. This means that there is no physical meaning in a charge conjugation of the mesons without a simultaneous strangeness conjugation.

We are now in a position to give a physical definition to the operations we are looking for:

1) We shall define as boson conjugation the simultaneous reversal of the sign of electric charge and strangeness of the whole meson field without touching anything concerning the bare baryon B_0: Then K transforms to $\bar{K}$ and our formulas (1) become:

$$
(2)\qquad
\begin{aligned}
&\Xi_0 = B_0 + K\,, && \Xi_0^- = B_0 + \overline{K^-}\,, && \Xi_0^0 = B_0 + \overline{K^0}\,,\\
&\Xi \;= \Xi_0 + \pi\,, && \Xi^- = \Xi_0^0 + \pi^- = \Xi_0^- + \pi^0\,, && \Xi^- = \Xi_0^- + \pi^+ = \Xi_0^0 + \pi^0\,.
\end{aligned}
$$

The states in which the N_0 and N particles transform are here defined as Ξ_0 and Ξ and it may be easily recognized, if we neglect mass differences, that they in effect possess the right values for charge and strangeness in order to identify them with the experimental Ξ's.

([2]) N. DALLAPORTA: *Nuovo Cimento*, **7**, 200 (1958).

([3]) J. SCHWINGER: *Phys. Rev.*, **104**, 1164 (1956).

2) Let us next define as spinor conjugation the transformation of the bare baryon B_0 into its antiparticle, without changing anything to the whole of its meson field. B_0 changes then into B_0, and formulas (1) become:

$$
\begin{aligned}
&\overline{X_0} = \overline{B_0} + K\,, \quad \overline{X_0^+} = \overline{B_0} + K^+\,, && \overline{X_0^0} = \overline{B_0} + K^0\,, \\
&\overline{X} = \overline{X_0} + \pi\,, \quad \overline{X^-} = \overline{X_0^0} + \pi^+ = \overline{X_0^+} + \pi^0\,, && \overline{X^0} = \overline{X_0^+} + \pi^- = \overline{X_0^0} + \pi^0\,.
\end{aligned} \tag{3}
$$

The new antistates obtained by this transformation are given the symbols $\overline{X_0}$ and $\overline{X}$. We shall identify them a little further.

3) Let us combine now the two B and S conjugations, in order to reverse simultaneously the signs of charge and strangeness of the whole meson field, and to transform the bare baryon into its antiparticle. What we obtain then, is obviously what is usually called charge conjugation and the transformed states for N_0 and N are:

$$
\begin{aligned}
&\overline{N_0} = \overline{B_0} + \overline{K}\,, \quad \overline{N_0} = \overline{B_0} + \overline{K}\,, && \overline{N_0^0} = \overline{B_0} + \overline{K^0}\,, \\
&\overline{N} = \overline{N_0} + \pi\,, \quad \overline{P^-} = \overline{N_0^-} + \pi^0 = \overline{N_0^0} + \pi^-\,, && \overline{N^0} = \overline{N_0^-} + \pi^+ = \overline{N_0^0} + \pi^0\,.
\end{aligned} \tag{4}
$$

$\overline{N_0}$ and $\overline{N}$ are now obviously the antinucleons. Moreover, the $C = B \cdot S$ transformation which enables us to go from (1) to (4), if now applied to (2) will give us the states defined by (3). This shows that the relation between the $\overline{X}$ and $\overline{X_0}$ to the Ξ and Ξ_0, is the same as the relation between the antinucleons $\overline{N_0}$ and $\overline{N}$ to the nucleons N_0 and N. This therefore enables us to define the $\overline{X_0}$ and $\overline{X}$ as the antiparticles of Ξ_0 and Ξ and allows us to give them the symbols $\overline{X_0} = \overline{\Xi_0}$, $\overline{X^+} = \overline{\Xi^+}$.

3. – We shall now give the mathematical expression for the transformation we are considering. Let is introduce the following symbols for the particle states:

$$
N_0 = \begin{pmatrix} p^0 \\ n_0 \\ \Xi_0^0 \\ \Xi_0^- \end{pmatrix}, \quad N = \begin{pmatrix} p \\ n \\ \Xi^0 \\ \Xi^- \end{pmatrix}, \quad K = \begin{pmatrix} K^+ \\ K^0 \\ \overline{K^0} \\ K^- \end{pmatrix} \tag{5}
$$

and define

$$
u = \begin{pmatrix} 1 & 0 \\ 0 & \pm i\tau_3 \end{pmatrix}.
$$

Then the Lagrangian of our model will be assumed as

$$(6)\qquad \begin{cases} L = L^0 + L' , \\ L' = \int \mathcal{L}' \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z , \\ \mathcal{L}' = g_\pi i\overline{\mathrm{N}}\gamma_5\tau_i\mathrm{N}_0\pi_i + g_k\overline{\mathrm{N}}_0 u\mathrm{K}(\Gamma)\mathrm{B}_0 + \text{h. c.} + \delta m_{\mathrm{N}}\overline{\mathrm{N}}\mathrm{N} + \delta m_{\mathrm{N}_0}\overline{\mathrm{N}}_0\mathrm{N}_0 . \end{cases}$$

The operation Γ means either $i\gamma_5$ or 1 according as we assume pseudoscalar or scalar coupling; we have chosen a representation for which $\gamma_5^2 = 1$; g_π and g_k are the pion and the interaction constants.

Then the transformations defining the B, the S and the C conjugation are given as follows:

$$(7)\qquad \left\{ \begin{array}{llll} & \mathrm{B} & \mathrm{S} & \mathrm{C} \\ \pi_i' & \pi_i(-1)^i & -\pi_i & \pi_i(-1)^{i+1} \\ \mathrm{K}' & -\overline{\mathrm{K}}^\Gamma & -\mathrm{K} & \overline{\mathrm{K}}^\mathrm{T} \\ \mathrm{B}_0' & -\mathrm{B}_0 & -\mathrm{C}^{-1}\mathrm{B}_0^\mathrm{T} & \mathrm{C}^{-1}\mathrm{B}_0^\mathrm{T} \\ \mathrm{N}_0' & \mathrm{BN}_0 & \mathrm{S}^{-1}\overline{\mathrm{N}}^\mathrm{T} & \mathbf{C}^{-1}\overline{\mathrm{N}}_0^\mathrm{T} \\ \mathrm{N}' & \mathrm{BN} & \mathrm{S}^{-1}\overline{\mathrm{N}}^\mathrm{T} & \mathbf{C}^{-1}\overline{\mathrm{N}}^\mathrm{T} \end{array} \right.$$

where $\mathrm{C}\gamma_\mu\mathrm{C}^{-1} = -\gamma_\mu^\mathrm{T}$, $\mathrm{B} = \tau_2^\mathrm{T}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, $\mathrm{S} = \mathrm{C}u_s$ $u_s = \tau_2^\mathrm{T}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$.

$$(8)\qquad \mathbf{C} = \mathrm{C}u_c , \qquad u_c = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) , \qquad \mathrm{B}\cdot\mathrm{S} = -\mathbf{C} .$$

One may easily verify that the Lagrangian (6) is invariant for all three operations B, S, and C given according to the transformations (7).

4. – The previous model or any other of the same kind that distinguishes the bare baryon from the mesonic field was necessary in order to allow us to give a precise physical meaning to the different conjugation states of the physical baryons that are obtained by applying the different operations we have been considering. However, now that these new physical states have been obtained according to the foregoing physical insight, we may use them in order to extend our results and apply our operation not only to a given baryon model, but to a real assembly of physical baryons of π and K mesons. If we assume the usually adopted interaction terms [4] for the different particles, we obtain a general interaction lagrangian:

$$(9)\qquad \begin{cases} L' = \int \mathcal{L}' \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z , \\ \mathcal{L}' = g_\pi i\overline{\mathrm{N}}\gamma_5\tau_i\mathrm{N}\pi_i + g_{\Sigma\Lambda}[\overline{\Lambda}(\Gamma)\boldsymbol{\pi}\cdot\boldsymbol{\Sigma} + \text{h. c.}] + \\ + ig_{\Sigma\Sigma}\sum_i \gamma_5 \sum_j \pi_k\varepsilon_{ijk} + g_{k\Lambda}[\overline{\mathrm{N}}u\mathrm{K}(\Gamma)\Lambda + \text{h. c.}] + g_{k\Sigma}[\overline{\mathrm{N}}(\Gamma)\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}\, u\mathrm{K} + \text{h. c.}] . \end{cases}$$

[4] B. D'ESPAGNAT and J. PRENTKI: *Nuclear Physics*, **1**, 33 (1956).

ε_{ijk} is the unitary antisymmetric tensor; the interaction constants, if expressed with those defined in the work of D'ESPAGNAT, PRENTKI and SALAM [5] are;

$$g_\pi = g_i = g_4\,,\quad g_{\Lambda\Sigma} = -g_2\,,\quad g_{\Sigma\Sigma} = -g_3\,,\quad g_{k\Lambda} = g_5 = g_7\,,\quad g_{k\Sigma} = g_6 = \pm g_8\,.$$

It is now easy to show that with the addition to table (7) of some new operations concerning the particles that were not considered as yet tabulated in table (10)

$$(10)\qquad \begin{cases} \begin{array}{llll} & \mathrm{B} & \mathrm{S} & \mathrm{C} \\ \Lambda' & -\Lambda & -C^{-1}\bar{\Lambda}^{T} & C^{-1}\bar{\Lambda}^{T} \\ \Sigma'_i & \Sigma_i(-1)^{i+1} & C^{-1}\bar{\Sigma}_i^{T} & C^{-1}\bar{\Sigma}_i^{T}(-1)^{i+1} \end{array} \end{cases}$$

one can extend the results of Sect. **3** to this more general Lagrangian and verify that it is in fact invariant for the three kinds of transformations (B, S and C conjugation) we have considered. The conclusion is then that strong pion and K interactions, which are the only ones included in our Lagrangian are invariant under the B and S operation separately, and as charge conjugation expresses the relation between e^+ and e^-, the three B, S and C conjugations may establish the connection between the four state $\mathrm{N}\Xi\bar{\Xi}\bar{\mathrm{N}}$.

5. – Let us now introduce in our Lagrangian further terms corresponding to the interaction with an electromagnetic field: we have of course to select only charged components and moreover the sign of the Ξ components of N is now reversed in respect to the sign of the nucleons; therefore our previous symbols (5) cannot be used any more and we must write with usual notations:

$$(11)\qquad \begin{cases} \mathcal{L}' = ie[\mathrm{K}^+ \operatorname{grad}_\mu \mathrm{K}^- - \operatorname{grad}_\mu \mathrm{K}^+\mathrm{K}^-]\cdot \mathrm{A}_\mu + \\ + ie(\pi^+ \operatorname{grad}_\mu \pi^- - \operatorname{grad}_\mu \pi^+\pi^-)\mathrm{A}_\mu - e^2(\mathrm{K}^+\mathrm{K}^- + \pi^+\pi^-)\mathrm{A}_\mu^2 + e\bar{\mathrm{P}}\gamma_\mu \mathrm{P}\cdot \mathrm{A}_\mu - \\ - e\bar{\Xi}^-\gamma_\mu \Xi^- \mathrm{A}_\mu + e\,\overline{\Sigma^+}\gamma_\mu \Sigma^+ \mathrm{A}_\mu - e\,\overline{\Sigma^-}\gamma_\mu \Sigma^- \mathrm{A}_\mu\,. \end{cases}$$

Let us investigate the invariance properties of these terms with respect to the same three kinds of transformations. It can then be verified as before that if we assume for A the following transformation properties:

$$(12)\qquad \begin{array}{llll} & \mathrm{B} & \mathrm{S} & \mathrm{C} \\ \mathrm{A}' & -\mathrm{A} & \mathrm{A} & -\mathrm{A} \end{array}$$

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[5] B. D'ESPAGNAT, J. PRENTKI and A. SALAM: *Nuclear Physics*, **3**, 446 (1957).

also all these new electromagnetic interaction terms remain invariant under the three B, S and C transformations. We can therefore generalize our previous statement and conclude that boson and spinor invariance are valid for all the strong and electromagnetic interaction terms generally considered in the usual hamiltonians.

6. – All the previous derivations have been obtained according to the assumption that the masses of all baryons are equal, and especially the masses of nucleons and Ξ's. This of course is not true, and the very conspicuous mass difference of these two particles indicates that there is a rather strong disturbance which prevents the B and S invariance properties to be effective. The reason for this dissymetry has been generally attributed to a different strength in the coupling for particles of opposed strangeness. As an example in the first Schwinger ([3]) scheme, it was supposed that the pion field was generated both by nuclear charge and hypercharge, and as the pairs of these quantum numbers were different for nucleons and Ξ's, it turned out that the total strength of the pion field generated by each of them, was different; in the Gell-Mann ([6]) scheme, it was explicitly supposed that the coupling constant g_{NKY} and $g_{\Xi\text{KY}}$ are different.

The consideration outlined in the present paper may perhaps suggest that the reason of the mass differences of the baryons would be accounted for by the introduction into the hamiltonian of one or more terms which, in order to preserve the equality of the masses of particles and antiparticles as supported by the properties of antinucleons, should preserve invariance under charge conjugation, but not under B and S conjugation separately.

And as the mass splitting is obviously connected with the strangeness dependent interactions, it would be natural to postulate that such a term should connect the baryons with the K mesons. A simples choice for it could be:

$$\bar{\mathrm{B}}\gamma_{\mu}\left(\bar{\mathrm{K}}\,\frac{\delta \mathrm{K}}{\delta x\mu}-\frac{\delta \bar{\mathrm{K}}}{\delta x\mu}\,\mathrm{K}\right)$$

where by B we intend any kind of baryon (B_0, N_0, N for the Lagrangian (6) or N, Λ, Σ, for Lagrangian (9) and now $\mathrm{K}=\begin{vmatrix}\mathrm{K}^+\\ \mathrm{K}^0\end{vmatrix}$ and $\bar{\mathrm{K}}=|\overline{\mathrm{K}^-}\;\overline{\mathrm{K}^0}|$.

It may be shown that such an interaction changes sign for both B and S conjugation and remains invariant for charge conjugation. According to it a baryon can emit and absorb in a single act two different K-mesons. If we add

([6]) M. Gell-Mann: *Phys. Rev.*, **106**, 1296 (1957).

such terms to our Lagrangians, one may see that self mass terms corresponding to diagrams such as:

Fig. 1.

change sign when we transform nucleons into Ξ's or anti Ξ's, while the sign remains the same if nucleons are transformed into antinucleons. Therefore, one would expect that the mass of both Ξ and anti Ξ should be shifted in respect to the masses of nucleons and antinucleons, as is in fact observed at least for nucleons and Ξ's.

RIASSUNTO

Se consideriamo i barioni come composti da un barione nudo senza carica nè spin e dai soliti campi mesonici (sia π che K), si può esaminare il comportamento del lagrangiano che descrive un tale sistema nei riguardi di una operazione che inverta i segni della carica e della stranezza dei mesoni senza toccare il barione nudo (coniugazione bosonica B) e di una operazione che trasformi il barione nudo nella propria antiparticella senza toccare i campi mesonici (coniugazione spinoriale S); il prodotto di tali due operazioni essendo evidentemente uguale alla solita coniugazione di carica C. Si trova che quando si trascurano le differenze di massa tra nucleone ed iperoni, tale lagrangiano è invariante per tutte e tre le operazioni B, S e C prese separatamente. La stessa proprietà si può pure dimostrare per il lagrangiano di d'Espagnat e Prentki di un insieme di nucleoni, iperoni e mesoni reali e per le interazioni di tali particelle col campo elettromagnetico. Infine viene prospettato che la differenza di massa tra nucleone e la Ξ possa essere dovuta ad un termine di interazione che sia invariante rispetto a C ma non rispetto a B ed S separatamente e si propone una semplice espressione che soddisfi a tali proprietà.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Analysis of Three Hypertriton Decays.

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(ricevuto il 25 Febbraio 1958)

Following the discovery of a hyperfragment by Danisz and Pniewski [1], a lot of hyperfragment decays have been reported. These investigations have brought light to hyperfragment.

In the course of a systematic scanning of hyperfragment in a stack of 7.5 cm $\times$ $\times$ 10 cm $\times$ 600 μm, 36 sheets of Ilford G5 emulsion pellicle exposed to a beam of 4.3 GeV π^--mesons from the Bevatron, a number of hyperfragment decays were found by area scanning. The three of them are interpreted as the mesonic decay of hydrogen hypernucleus ${}^3H_\Lambda$ of which we are going to report here.

The details of the decay events are illustrated in the first to the fifth columns in Table I.

The identification of the particles which produced the tracks in question was made by the δ-ray count, the gap count, the scattering measurement, and by the visual comparison of the tracks with those produced by the known particles in the vicinity. The identification of the two π^--meson tracks stopped in the emulsion was confirmed by the characteristic σ-star at the stopping, and also that of the other π^--meson track which went out of the stack was made by the change in the ionization along the track. The evaluation of the stopping power of the emulsion was made by the mean range of (607.5 ± 3.4) μm of nine flat μ-mesons of 4.12 MeV from π-μ decay.

The energies were evaluated by the range-energy curve of Barkas *et al.* to the π^--mesons and by that of Baroni *et al.* to the others.

All the three decay events were established as the decay of ${}^3H_\Lambda$ by the following scheme:

$${}^3H_\Lambda \to \pi^- + p + p + n + Q\,.$$

All the other possible interpretations of the decays, for example, ${}^4H_\Lambda \to \pi^- + p + p + 2n$, $\to \pi^- + d + p + n$, $\to \pi^- + t + p$, $\to \pi^- + d + d$, etc., were excluded because in those cases, the experimental binding energy of Λ^0-particle in hypernucleus become unacceptable negative values.

The information obtained from these decays of ${}^3H_\Lambda$ fragment, is given in the seventh column to the fifteenth column: the time of flight of the hypertriton, momentum of the decay particles, Q-value of the decay, the binding energy, the momentum and the energy of Λ^0-particle in hypertriton, and the angle in the Λ^0 center-of-mass system between the direction of the Λ^0-meson from the Λ^0

[1] M. Danysz and J. Pniewski: *Phil. Mag.*, **44**, 384 (1953).

TABLE I. – *Characteristics of the hypertritons and their decay.*

Event	Parent star	Track	Range (μm)	Identity	Energy (MeV)	Time of flight ($\cdot 10^{-11}$ s)	Component of momentum (MeV/c) p_x	p_y	p_z	Q (MeV)	B_Λ (MeV)	P_Λ (MeV/c)	E_Λ (MeV)	$\theta_{\Lambda\pi}$ (deg.)	$\cos\theta_{\Lambda\pi}$
		H.F.	224.1	$^3H_\Lambda$	9.1	1.4				35.7	-1.0 ± 0.6	113.7 ± 7.0	5.8 ± 0.6	154	-0.91
1	$12+3\pi^-$	1	4.8	p	0.44		28.94	0.	0.						
		2	17259.0	π^-	30.82		-96.09	-17.29	-5.44						
		3	4.9	p	0.46		-11.41	-15.76	21.79						
		H.F.	226.4	$^3H_\Lambda$	9.1	1.4				34.7	0.0 ± 0.6	63.5 ± 6.5	1.8 ± 0.6	11	0.98
		1	338.5	p	7.43		114.58	0.	-29.09						
2	$23+5\pi^-$	2	2.7	p	0.27		-24.67	10.21	0.						
		3	12241.0	π^-	25.66		-78.76	-39.27	-8.51						
		H.F.	458.0	$^3H_\Lambda$	14.1	2.2				34.5	0.5 ± 4.3	121.8 ± 30.0	7.4 ± 4.3	23	0.92
		1	105.4	p	3.70		80.73	0.	-19.12						
3	$6+4\pi^-$	2	4.8	p	0.45		-24.63	-15.04	2.43						
		3	$>7000.$	π^-	24.5 ± 3.6		25.87	65.35	49.93						

decay and the line of flight of the Λ^0 particle (2).

* * *

We are very grateful to Professor E. J. LOFGREN who arranged the exposure of the stack and to Dr. W. W. CHUPP and Dr. T. KOTANI who exposed the stack in perfect conditions and supplied us with the relevant data and to Professors T. YASAKI and H. MORI for their help.

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(2) P. ZIELINSKI: *Nuovo Cimento*, **3**, 1179 (1956).

2148

On Hall's Formula for the Relativistic Photoeffect (*).

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(ricevuto il 19 Aprile 1958)

H. Hall has succeeded in obtaining for the extreme relativistic, K-shell photoeffect cross-section, the following formula ([1-3])

$$\sigma_K = \sigma_K^0 \frac{1-(2\delta/3)}{1-(\delta/2)} \frac{\exp\left[-\pi\alpha Z + 2\alpha Z \operatorname{arctg}(\alpha Z/\gamma)\right]}{(\alpha Z)^{2\delta}} J(\alpha Z)\,, \tag{1}$$

where notations similar to those of Hall have been used, *i.e.*

$$\sigma_k^0 = 4\pi a_0^2 \alpha^8 Z^5 \sqrt{1-\beta^2}\,, \qquad \gamma = [1-(\alpha Z)^2]^{\frac{1}{2}}\,, \qquad \delta = 1-\gamma\,, \tag{2a}$$

$$J(\alpha Z) = \frac{3}{8}\int_0^\infty\!\!\int_0^\infty du\,dv\, \frac{[\mathcal{A}]^{-(\delta/2)}}{(u+u^2)^{\frac{1}{2}}(v+v^2)^{\frac{1}{2}}(1+u+v)^{4-2\delta}\{\mathcal{B}\}^{-1}}\,, \tag{2b}$$

$$[\mathcal{A}] = \left(\cos^2\frac{b}{2}+u\right)\left(4\sin^2\frac{b}{2}+4u\right)\left(\cos^2\frac{b}{2}+v\right)\left(4\sin^2\frac{b}{2}+4v\right), \tag{2c}$$

$$\{\mathcal{B}\} = \left\{\cos\varphi(u)\cos\varphi(v) + \left[(1+2u)\cos\varphi(u) - 2(u+u^2)^{\frac{1}{2}}\operatorname{cotg}\frac{b}{2}\sin\varphi(u)\right]\cdot \right.$$
$$\left.\cdot\left[(1+2v)\cos\varphi(v) - 2(v+v^2)^{\frac{1}{2}}\operatorname{cotg}\frac{b}{2}\sin\varphi(v)\right]\right\}, \tag{2d}$$

$$\varphi(u) = \alpha Z \ln\left(1+2u+2\sqrt{u+u^2}\right) - \delta \operatorname{arctg}\frac{2\gamma}{\alpha Z}\frac{\sqrt{u+u^2}}{1+2u}\,; \quad \sin b = \alpha Z\,. \tag{2e}$$

(*) A more detailed discussion of the problem will be published in: *Stud. Cercet. Fiz.* (1958), (in Rumanian).

(1) H. Hall: *Rev. Mod. Phys.*, **8**, 358 (1936), the formula for $\tau_k = N_0\sigma_k$ on p. 395.

(2) H. Hall: *Phys. Rev.*, **84**, 167 (1951).

(3) A recent, more simple demonstration of this result was given by R. Prange and R. Pratt: *Phys. Rev.*, **108**, 139 (1957).

Formula (1) is exact in αZ, for the limit $\beta \to 1$, in which only the lowest power of $(1-\beta^2)$ is retained.

As written in (1) the cross-section is practically useless, because of the unevaluated integral $J(\alpha Z)$ it contains. In order to find an approximate expression for it. HALL proceeds as follows. *a*) Sets u and v equal to $1/7$ in the δ power factors of the integrand. *b*) Keeps only the zero and first order terms of the expansion of $\{\mathcal{B}\}$ in powers of αZ. *c*) Sets again $u=v=1/7$ in the arguments of the arctg and ln functions, which appear in the *first order* term of the expansion of $\{\mathcal{B}\}$ [4]. He thus finds for $J(\alpha Z)$ the expression

$$J_H = \left|\frac{(1+2/7)^2}{(1+1/7)(\alpha^2Z^2+4/7)}\right|^{\delta}\left(1+0.05 \operatorname{arctg}\frac{0.63}{\alpha Z}\right). \tag{3}$$

Taking this result to be good, its substitution into (1) yields an expression for σ_K which, as shown by HALL, is well approximated by

$$\sigma_K^H = \sigma_K^0 H(\alpha Z)\,; \qquad H(\alpha Z) = \exp\left[-\pi\alpha Z + 2\alpha^2Z^2(1-\ln\alpha Z)\right]. \tag{4}$$

Even in the extreme case of Pb ($\alpha Z=0.60$), the use of Eq. (4) instead of (1) combined with (3), does not introduce a difference greater than 4%. Formula (4) is frequently quoted in literature [5].

Actually, as we shall presently show, Hall's approximation *c*) is wrong, his expression (3) being a bad approximation for $J(\alpha Z)$ and hence formula (4) unjustified. To this end we will evaluate the integral correctly, to first order in αZ.

Expanding in powers of αZ the quantities involved in the integrand of $J(\alpha Z)$ and denoting the terms of order $(\alpha Z)^K$ by O_K, we find from (2)

$$\delta = \frac{1}{2}(\alpha Z)^2 + O_4\,, \qquad \operatorname{cotg}\frac{b}{2} = \frac{2}{\alpha Z} + O_1\,, \qquad \cos\varphi(u) = 1 + O_2\,,$$

$$\operatorname{cotg}\frac{b}{2}\sin\varphi(u) = 2\ln\left(1+2u+2\sqrt{u+u^2}\right) - \alpha Z \operatorname{arctg}\frac{2\gamma}{\alpha Z}\frac{\sqrt{u+u^2}}{1+2u} + O_2\,.$$

Hence

$$\begin{aligned}\{\mathcal{B}\} &= \{\mathcal{B}\}_0 + \\ &+ 2\alpha Z\left\{(u+u^2)^{\frac12}\left|\operatorname{arctg}\frac{2\gamma}{\alpha Z}\frac{\sqrt{u+u^2}}{1+2u}\right|\left[1+2v-4(v+v^2)^{\frac12}\ln\left(1+2v+2\sqrt{v+v^2}\right)\right] + \right.\\ &\left. + (v+v^2)^{\frac12}\left|\operatorname{arctg}\frac{2\gamma}{\alpha Z}\frac{\sqrt{v+v^2}}{1+2v}\right|\left[1+2u-4(u+u^2)^{\frac12}\ln\left(1+2u+2\sqrt{u+u^2}\right)\right]\right\} + O_2\,,\end{aligned}$$

where $\{\mathcal{B}\}_0$ is the zero order term of $\{\mathcal{B}\}$, considered also by HALL. We may write therefore

$$J(\alpha Z) = J_0 + \tfrac{3}{2}\alpha Z J_1' + O_2\,, \tag{5}$$

[4] The explanation of these approximations is given in ref. (1), p. 396.

[5] H. BETHE and E. SALPETER: *Encyclopedia of Physics*, **35**, Part I (Berlin, 1957), Sect. **73**; W. HEITLER: *The Quantum Theory of Radiation* (Oxford, 1954), p. 210, etc.

where J_0 is the value of $J(\alpha Z)$ for $\alpha Z = 0$, and

$$(6)\qquad J_1' = \int_0^\infty\!\!\int_0^\infty du\,dv\,\frac{\left(\operatorname{arctg}\dfrac{2\gamma}{\alpha Z}\dfrac{\sqrt{u+u^2}}{1+2u}\right)\left[1+2v-4(v+v^2)^{\frac{1}{2}}\ln\left(1+2v+2\sqrt{v+v^2}\right)\right]}{(v+v^2)^{\frac{1}{2}}(1+u+v)^4}\,.$$

J_1' is still αZ dependent through the argument of the arctg function. However, it may be shown (*), that by substituting $\pi/2$ for the arctg factor, the error thereby introduced is of first order in αZ for J_1' and hence of second order for $J(\alpha Z)$. We may thus write

$$(7)\qquad J(\alpha Z) = J_0 + \tfrac{3}{4}\pi\alpha Z J_1 + O_2\,,$$

where

$$(8)\qquad J_1 = \int_0^\infty\!\!\int_0^\infty du\,dv\,\frac{1+2v-4(v+v^2)^{\frac{1}{2}}\ln\left(1+2v+2\sqrt{v+v^2}\right)}{(v+v^2)^{\frac{1}{2}}(1+u+v)^4}\,.$$

The integral J_0 has been evaluated by Hall, to give 1 [6]. The u integration in (8) is immediate and yields

$$(9)\qquad \begin{cases} J_1 = \frac{1}{3}(P-Q)\,; \\[6pt] P = \displaystyle\int_0^\infty \frac{1+2v}{(v+v^2)^{\frac{1}{2}}(1+v)^3}\,dv\,, \quad Q = \int_0^\infty \frac{\ln\left(1+2v+2\sqrt{v+v^2}\right)}{(1+v)^3}\,dv\,. \end{cases}$$

Performing the v integrations we find $P = 8/5$; $Q = 8/3$; $J_1 = -16/45$. Thus according to (7) the expression of $J(\alpha Z)$, correct to first order in αZ, becomes [7]

$$(10)\qquad J(\alpha Z) = 1 - \frac{8}{15}\frac{\pi}{2}\alpha Z + O_2\,.$$

(The use of Hall's approximation *c*) in the evaluation of the integral Q of Eq. (9), yields to first order the result (3), as expected.)

The comparison of (3) with the correct result (10) leads to the conclusion that, disregarding the second order terms O_2, J_H is a bad approximation for $J(\alpha Z)$. Indeed the difference $J_H - J(\alpha Z) = 0.90\,\alpha Z$ shows that J_H is in error with about 32% for Sn and 54% for Pb. As to the O_2 corrective terms, they are numerous and the questionable possibility of approximating some of them by means of *a*) is useless, so long as nothing is known about the others. Thus Hall's conclusions, being based essentially upon (3), are wrong. Formula (4) is a delusive description of the extreme relativistic cross-section (1), since the errors it involves are of the same order of magnitude as the last two terms contained in $H(\alpha Z)$.

(6) Ref. (1); J_0 is the integral of Eq. (44) multiplied by 3/8.

(7) Prange and Pratt [ref. (3)] succeeded in evaluating $J(\alpha Z)$ exactly, in the special case (not realized physically) $\alpha Z = 1$, finding $J(1) = 0{,}24$. Our approximation (10) gives in this case (falling far out of its range of validity) the value 0.17. This good agreement could eventually suggest that the rest O_2, of Eq. (10), is indeed negligible for ordinary αZ.

Combining (1) and (10) we find that the correct expression for σ_K is given to first order by

$$\sigma_K = \sigma_K^0 \left(1 - \pi\alpha Z - \frac{4}{15}\pi\alpha Z + O_2\right). \tag{11}$$

This result may be derived also by an entirely different method [8,9].

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[8] By using a higher order Born approximation for the final state spinor of the electron, the present author was enabled to find the αZ corrective terms to Sauter's cross section (to be published). His result reduces to (11) in the extreme relativistic limit.

[9] The curves of HULME *et al.* (*Proc. Roy. Soc.* A **149**, 131, (1935); Fig. 2), representing $5\cdot 10^{32}\,\sigma_K\, h\nu/4Z^5\, mc^2$ as function of $mc^2/h\nu$ for different Z, are based in the extreme relativistic limit on the formula of HALL (4), which, as shown, is in error by excess. These curves should be therefore adequately lowered in the energy range: $0<(mc^2/h\nu)<0{,}45$.

The High-Energy End of the Electron-Bremsstrahlung Spectrum.

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(ricevuto il 23 Aprile 1958)

Recent measurements of the spectrum of betatron produced bremsstrahlung furnished some information about the shape of the spectrum at its high-energy end. The results of Fuller *et al.* (1) indicate that the intensity at the tip of the spectrum is higher than predicted by the Bethe-Heitler formula. Since in this limit Born's approximation is incorrect (2) the discrepancy is not unexpected. The following calculation, based upon Sommerfeld-Maue's approximation, will indeed lead to a non-zero limit of the cross-section for $k \to E_0 - mc^2$. The results are expected to be satisfactory except for heavy nuclei.

The matrix element for bremsstrahlung is (2)

$$H'_{12} = -e\hbar c\left(\frac{2\pi}{k}\right)^{\frac{1}{2}}\int \psi^*_{f,-}\alpha_\lambda \exp\left[-i\boldsymbol{k}\boldsymbol{r}\right]\psi_{i,+}\,\mathrm{d}\tau\,,$$

where α_λ is the component of the Dirac matrix operator $\boldsymbol{\alpha}$ in the direction of the photon polarization. The subscripts $+$ and $-$ refer to wave functions with asymptotic forms: plane wave plus outgoing or ingoing spherical wave, respectively.

Energy-momentum considerations show that the momentum transferred to the nucleus, $\boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{k}$ is of the order 1 (in mc units), for small angles between $\boldsymbol{p}_1$ and $\boldsymbol{k}$. In this case screening cannot have much influence and wave functions for a Coulomb potential should lead to a satisfactory accuracy. Sommerfeld-Maue's approximation (3) for these functions written in Bethe-Maximon's notation (B.M.) are

$$\psi_{i,+} = N_1 \exp\left[-i\boldsymbol{p}_1\boldsymbol{r}\right]\left(u_1\,\Phi_1 - \frac{i}{2\varepsilon_1}\boldsymbol{\alpha}\nabla\,\Phi_1 u_1\right),$$

$$\psi^*_{f,-} = N^*_2 \exp\left[-i\boldsymbol{p}_2\boldsymbol{r}\right]u^*_2\left(\Phi_2 + \frac{i}{2\varepsilon_2}\boldsymbol{\alpha}\nabla\,\Phi_2\right),$$

(1) E. G. Fuller, E. Hayward and H. W. Koch: *Phys. Rev.*, **109**, 630 (1958).

(2) W. Heitler: *The Quantum Theory of Radiation* (Oxford, 1954), p. 242.

(3) A. Sommerfeld and A. W. Maue: *Ann. Phys.*, **22**, 629 (1935); H. A. Bethe and L. C. Maximon: *Phys. Rev.*, **93**, 768 (1953). Referred to later as (B.M.).

where N_{12} are normalization constants, $u_{1\,2} = u(\boldsymbol{p}_{1,2})$ the normalized Dirac matrix coefficients for free electrons, and $\Phi_{1\,2}$ the confluent hypergeometric function,

$$\Phi_1 = \Phi(ia\varepsilon_1/p_1,\ 1,\ ip_1 r - i\boldsymbol{p}_1\boldsymbol{r}),$$

$$\Phi_2 = \Phi(ia\varepsilon_2/p_2,\ 1,\ ip_2 r + i\boldsymbol{p}_2\boldsymbol{r}).$$

The differential cross-section for the process is (B.M.)

$$\mathrm{d}\sigma = \frac{(2\pi)^2}{\hbar c}(mc^2)^4 \frac{p_2}{p_1}\frac{\varepsilon_1\varepsilon_2}{(2\pi\hbar c)^6}\sum |H'_{12}|^2 k^2\,\mathrm{d}k \sin\theta_1\,\mathrm{d}\theta_1 \sin\theta_2\,\mathrm{d}\theta_2\,\mathrm{d}\varphi\,,$$

where the sum is intended over all polarization states of the photon and the electrons. An estimation of the integrals and matrix factors in $\sum|H'_{12}|^2$ shows that the contributions from the gradient term in the function $\psi_{i,+}$ is of relative order $1/\varepsilon_1$ and therefore negligible.

In the limit $p_2 \to 0$ we have $\sum|H'_{12}|^2 = 0(1/p_2)$ because of

$$|N_2|^2 = \pi a_2 \exp[\pi a_2]/(\sinh \pi a_2)\,.$$

This leads to a non-zero value of $\mathrm{d}\sigma_0 = \lim\limits_{p_2\to 0}\{\mathrm{d}\sigma\}$. If $\varepsilon_1 \gg 1$ and if $\theta_1 \to 0$ (forward direction) the result is

$$\mathrm{d}\sigma_0 = \frac{(\pi a)^2}{\sinh \pi a}\, 2\exp[-\pi a]\frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^4 \frac{\varepsilon_1}{p_1}\frac{1}{2\pi}\, k\,\mathrm{d}k \sin\theta_1\,\mathrm{d}\theta_1 \sin\theta_2\,\mathrm{d}\theta_2\,\mathrm{d}\varphi\,,$$

$$2(1-\cos\theta_2)\{\Phi(1)\Phi^*(1) + (1+a^2)\Phi(2)\Phi^*(2) - [(1+ia)\Phi(1)\Phi^*(2) + (1-ia)\Phi^*(1)\Phi(2)]\}\,.$$

where

$$\Phi(1) = \Phi(1+ia,\ 1,\ ia(1-\cos\theta_2))\,,$$

$$\Phi(2) = \Phi(2+ia,\ 2,\ ia(1-\cos\theta_2))\,.$$

Integrating $\mathrm{d}\sigma_0$ over $\mathrm{d}\varphi$ and $\mathrm{d}\theta_2$ we obtain the differential cross-section σ_0, (for forward direction) per unit energy interval of the photon at the high-energy end. The result can be expressed as

$$\sigma'_0 = B\frac{Z^2}{137}\left(\frac{e^2}{mc^2}\right)^2 \varepsilon_1\,.$$

where B is a numerical factor. A few representative values of B, obtained by numerical integration, are quoted in the following table:

Element	B
Al	0.0161
Ni	0.111
Mo	0.267
W	0.649

The values of σ_0' appear low, since for $E_0 = 30$ MeV at about 50 keV below the tip Bethe-Heitler's formula leads to values of the same order of magnitude. It seems that the detailed shape of the end of the spectrum could be revealed experimentally only with small channel or resonance widths (< 50 keV).

In a subsequent paper the angular distribution of the bremstsrahlung and a detailed comparison with Bethe-Heitler's formula will be presented.

* * *

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A General Type of Experiments Proposed to Test the Conservation of Parity in Strong Interactions at High Energies.

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(ricevuto il 5 Maggio 1958)

1. – LEE and YANG ([1]) suggested recently the possibility, followed soon by experimental verification ([2-4]), that parity may not be conserved in weak interactions. For strong interactions, MORPURGO and TOUSCHEK ([5]) observe that no proof of the conservation of parity has been obtained, except for nuclear reactions ([6]) and for pion-nucleon interactions at low energies; they have shown that, assuming CP invariance, the conservation of parity is a consequence of the conservation of isotopic spin for the pion-nucleon interaction and of the gauge-invariance for the electromagnetic interaction. On the other hand, there is no such theoretical argument or experimental proof to stand in favour of the conservation of parity, in reactions producing strange particles and in reactions between pions and nucleons where virtual states containing strange particles may be more important ([7-9]). It is therefore

([1]) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956).

([2]) C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPES and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957).

([3]) R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957).

([4]) J. I. FRIEDMANN and V. L. TELEGDI: *Phys. Rev.*, **105**, 1681 (1957).

([5]) G. MORPURGO and B. TOUSCHEK: private communication.

([6]) N. TANNER: *Phys. Rev.*, **107**, 1203 (1957). This experiment shows that parity is conserved in nuclear reactions at least to within 1 part in 10^7 in intensity. One may observe that contributions from strange particles intermediate states may possibly be felt already at such high precision; however, no precise quantitative statement seems possible at present.

([7]) This remark is due to R. GATTO: It is formally possible to relate conservation of parity to conservation of strangeness, except that one meets some difficulty in dealing with interactions such as (NΛK) where strangeness is also taken off by a boson. It may be remarked that in the scheme for strong interactions recently proposed by OKUN (to be published), in which the fundamental interactions are of the form (NN, NN) and (ΛΛ, NN), one can give a formal argument based on Tiomno's mass reversal which leads to conservation of parity.

([8]) It has been pointed out, however, that in Λ^0 production parity non conservation in strong interactions may lead to a longitudinal polarization of the Λ^0, which in turn will manifest itself through a forward-backward or left-righ-asymmetry in the subsequent Λ^0 decay. No such effects have been observed so far. See ref. ([9]).

([9]) F. EISLER, R. PLANO, A. PRODELL, N. SAMIOS, M. SCHWARTZ, J. STEINBERGER, P. BASSI, V. BORELLI, G. PUPPI, G. TANAKA, P. WALOSCHEK, V. ZOBOLI, M. CONVERSI, P. FRANZINI, I. MANELLI, R. SANTANGELO, V. SILVESTRINI, G. L. BROWN, D. A. GLASER and C. GRAVES: *Nuovo Cimento*, **7**, 222 (1958). The experiments of this type (observation of production and decay of hyperons) are also an important way to test conservation of parity in strong interactions.

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of interest to gain experimental data with the purpose to decide upon this point. The general type of experiments proposed in this letter may possibly lead to a decision; for some characteristic features, these experiments are probably easier to be carried out than polarization experiments. Indeed, a polarization experiment requires almost always the observation of two consecutive interactions, due to the difficulty to obtain a polarized high energy primary beam. The proposed experiments require only one production interaction and unpolarized primary beams.

2. – The experiments can be carried out by studying those reactions between elementary particles resulting in a final state in which three different particles are present. We shall now consider for instance the following reaction:

$$(1) \qquad \pi^- + p \to n + \pi^+ + \pi^- ,$$

which can be obtained by bombarding a target containing unpolarized hydrogen with a high energy π^- beam.

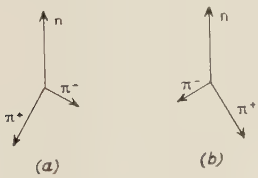

Fig. 1.

Fig. 1(*a*) shows the momenta of the three out-coming particles, projected on a plane (the plane of the sketch) which is *normal* to the line of flight of the incident π^- (that is the transverse momenta).

In Fig. 1(*b*) are shown the same transverse momenta as in Fig. 1(*a*), but in a different case, obtained from Fig. 1(*a*) through a reflection with respect to the plane which contains the directions of the neutron and of the incident π^-. If there is conservation of parity in the interaction, both cases (*a*) and (*b*) will have the same yield. Indeed, an asymmetric yield would enable definition of the handedness of a screw by associating the direction of the incident π^- with the direction of rotation (on the plane of Fig. 1) $n \to \pi^+ \to \pi^-$ favoured in the result of the experiment. Therefore a different yield of the two events will be possible only if the interaction does not conserve the parity.

The possible existence of such an asymmetry could be detected experimentally by means of any technique which records and identifies simultaneously the two out-coming π-mesons: for instance, the hydrogen bubble chamber, or the hydrogen filled diffusion chamber, in a magnetic field, or else two counters coupled to a magnetic system being able to identify the sign of the counted particles. There is no need for detecting simultaneously the neutron. The random coincidences between counters, due to the $\pi^- + p \to n + \pi^+ + \pi^- + \pi^0$ reaction, do not cause any trouble, since the experiment will detect at all events the asymmetry we are considering (Fig. 1(*a*) and Fig. 1(*b*)), with or without other associated mesons.

3. – Reaction (1) is only an example. The same considerations are valid for any reaction between elementary particles (unpolarized) resulting in a final state with three different out-going particles as well as for any reaction in which

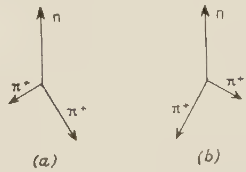

Fig. 2.

two of the three out-coming particles, although being identical, are very different in the momenta. The sketch of Fig. 2 may illustrate the case of the

reaction $\pi^+ + p \to \pi^+ + \pi^+ + n$ (the plane of the figure is normal to the incident π^+).

For the above mentioned reasons, the reactions

$$\pi^- + p \to n + K^0 + \bar{K}^0 \; ; \; \gamma + p \to p + K^+ + K^-$$

are among the most interesting ones.

The following three reactions should be easier to be observed experimentally, since they do not require either chambers nor magnets, but can be observed with counters alone, at least in principle:

$$\pi^- + p \to p + \pi^- + \pi^0$$

$$n + p \to n + p + \pi^0$$

$$\gamma + p \to n + \pi^+ + \pi^0$$

In these cases, there is the possibility to count the coincidences between one γ from π^0's decay and one charged outcoming particle.

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Heisenberg's Universal Theory and Gravitation.

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Recently Heisenberg [1] proposed a simple non-linear equation for a universal spinor field including all elementary particles as « excited states » of this field as well as their strong and electromagnetic interactions. It is not yet clear whether this equation includes also automatically weak interactions. Therefore let us assume the fundamental equation in the form

$$\text{(1)}\qquad \begin{cases} [\sigma^k\partial_k + L^2(...) + Ll(...)]\psi = 0 \\ [\tau^k\partial_k + L^2(...) + Ll(...)]\chi = 0 \end{cases},$$

where (...) denote some bilinear expressions in ψ, χ, $\psi^\dagger$, $\chi^\dagger$ whereas L and l are two universal lengths. If $\boldsymbol{L}$ is of the order 10^{-13} cm and l is about 10^{-7} times smaller, the first non-linear term will describe strong and electromagnetic interactions whereas the second can be made responsible for weak (Fermi) interactions with the coupling constant $\boldsymbol{g}^2 \sim 10^{-14}$. The field quantities ψ and χ are two two-component spinors with the transformation laws

$$\text{(2)}\qquad \psi' = \Lambda\psi\,, \qquad \chi' = \chi\Lambda^{-1}$$

in the spin space. The matrices σ^k and τ^k are

$$\text{(3)}\qquad \begin{aligned} \sigma^0 &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, & \sigma^1 &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\ \sigma^2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, & \sigma^3 &= \begin{pmatrix} & i \\ i & \end{pmatrix}, \end{aligned}$$

and

$$\tau^0 = \sigma^0\,, \quad \tau^1 = -\sigma^1\,, \quad \tau^2 = -\sigma^2\,, \quad \tau^3 = -\sigma^3\,.$$

The equations (1) obviously do not comprise gravitation. However, if Heisenberg's idea of a universal field is correct, the fundamental equation should comprise all material phenomena including gravitation. In other words, the universal field should determine also the metric properties of space-time. The question arises whether and how it is possible?

The equations (1) can be rewritten for the case of a riemannian geometry by the following replacements

$$\text{(4)}\qquad \sigma^k\partial_k\psi \to \sigma^{\boldsymbol{k}}\psi_{;\boldsymbol{k}}\,, \quad \tau^k\partial_k\chi \to \tau^{\boldsymbol{k}}\chi_{;\boldsymbol{k}}\,.$$

where $\sigma^{\boldsymbol{k}}$ and $\tau^{\boldsymbol{k}}$ are vectors with respect to the general co-ordinate transformations while $\psi_{;\boldsymbol{k}}$ and $\chi_{;\boldsymbol{k}}$ mean covariant derivatives ($\boldsymbol{k}$, $\boldsymbol{l}$, ... mean tensor indices for the case of a riemannian geometry in

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([1]) W. Heisenberg: *Rev. Mod. Phys.*, **29**, 3, 269 (1957).

contradistinction to $k, l, \ldots$ which can be regarded as tensor indices only under Lorentz transformations). Following the exposition of BERGMAN [2] (which is a specialization but also a simplification of the theory of Infeld and van der Waerden [3]) the quantities involved in (4) are

$$(5)\qquad \begin{cases} \boldsymbol{\sigma}^k = u^k\sigma^0 + \alpha^k\sigma^1 + \beta^k\sigma^2 + \gamma^k\sigma^3 \\ \boldsymbol{\tau}^k = u^k\tau^0 + \alpha^k\tau^1 + \beta^k\tau^2 + \gamma^k\tau^3 \end{cases},$$

$$(6)\qquad \begin{cases} \boldsymbol{g}^{kl} = -\tfrac{1}{2}(\boldsymbol{\tau}^k\boldsymbol{\sigma}^l + \boldsymbol{\tau}^l\boldsymbol{\sigma}^k) \\ \boldsymbol{g}_{kl}\,\boldsymbol{g}^{lm} = \delta_k{}^m \end{cases},$$

$$(7)\qquad \begin{cases} \psi_{;k} = (\partial_k + \Gamma_k)\psi \\ \Gamma_k = -\tfrac{1}{4}\boldsymbol{\tau}_l(\partial_k\boldsymbol{\sigma}^l + \Gamma^l_{km}\boldsymbol{\sigma}^m) \end{cases},$$

where $\Gamma^l_{km} = \Gamma^l_{mk}$ are Christoffel symbols of the second kind for the case of a metric defined by (6).

An external gravitational field is given if the sixteen real functions u^k, α^k, β^k, γ^k are given. However, since at least six of them are superfluous, we shall restrict ourselves to ten quantities

$$(8)\qquad f^{kl} = f^{lk}$$

and put

$$(9)\qquad \begin{cases} u^k = \delta^k{}_0 + f^k{}_0\,, & \alpha^k = \delta^k{}_1 + f^k{}_1\,, \\ \beta^k = \delta^k{}_2 + f^k{}_2\,, & \gamma^k = \delta^k{}_3 + f^k{}_3\,, \end{cases}$$

where

$$(10)\qquad f^k{}_l = f^{km}\eta_{ml}\,,$$

with

$$(11)\qquad \begin{cases} \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1 \\ \text{and} \\ \eta_{kl} = 0 \text{ for } k \neq l\,. \end{cases}$$

From (5) and (9)

$$(5')\qquad \boldsymbol{\sigma}^k = \sigma^k + f^k{}_l\sigma^l\,, \quad \boldsymbol{\tau}^k = \tau^k + f^k{}_l\tau^l\,.$$

From (6) and (5′) we get

$$(6')\qquad \boldsymbol{g}^{kl} = \eta^{kl} + 2f^{kl} + f^{ks}f^l{}_s\,.$$

Now, the metric tensor involves ten arbitrary functions as usually.

The decisive step is as follows: instead of regarding f^{kl} as given functions of x^k, assume them to depend on x^k through ψ and χ and their hermitian conjugates. A plausible form of f^{kl} is

$$(12)\qquad f^{kl} = l^6[(\psi^\dagger\sigma^k\psi)(\chi^\dagger\tau^l\chi) + (\psi^\dagger\sigma^l\psi)(\chi^\dagger\tau^k\chi)]\,,$$

where a sixth power of a constant l with dimension of length appears for dimensional reasons. Introducing (12) into (5′) and (6′) the equations (1) modified according to (4) become non-linear equations with higher degrees of non-linearity. The lowest non-linearities appearing in the new field equations and responsible for gravitational effects involve products of five field quantities multiplied by l^6. In order to get a correct order of magnitude of the gravitational coupling constant we do not need to assume a new constant l in (12) but can identify it with that responsible for weak interactions. Indeed, since weak interactions are described by a non-linear term proportional to l (which corresponds to a dimensionless constant $\boldsymbol{g}^2 \sim 10^{-14}$) and the gravitational interactions are described by a non-linear term proportional to l^6, the dimensionless gravitational constant $\boldsymbol{G}/\boldsymbol{hcL}^2$ must be equal to $\boldsymbol{g}^6 \sim 10^{-42}$ or 10^{-41}. This is just what is needed if, according to HEISENBERG, we assume the universal length $\boldsymbol{L}$ to be about 1/8 of the nucleonic Compton wave length. In this way a close connection between the gravitational and the weak coupling constant is made plausible.

The above idea of regarding the gravitational field as an effect of a self-action brought about by some non-linearities in the fundamental spinor field equation differs essentially from the or-

[2] P. G. BERGMAN: *Phys. Rev.*, **107**, 624 (1957).

[3] L. INFELD and B.L. VAN DER WAERDEN: *Sitzber. preuss. Akad. Wiss.* 380 (1933).

thodox views. First of all it is noticed that in this formulation there exist privileged frames of reference where the metric tensor assumes a particularly simple form (6'). This resembles the situation in the special theory of relativity where some frames of reference (inertial frames) are distinguished by a particularly simple form of the metric tensor, namely η_{kl}. Thus, those frames of reference where (6) with (12) hold will be called quasi-inertial frames of reference. Hereby the statements: « weak and strong gravitational fields » acquire an absolute meaning: the gravitational field is weak (strong) if it is weak (strong) in the quasi-inertial frame of reference. Most physicists do not believe in the existence of quasi-inertial frames of reference, but it should be stressed that the applicability of an approximation method, *e.g.* that of Einstein-Infeld-Hoffmann ([4]) could not be justified without assuming the existence of a privileged class of frames of reference. Therefere we do not regard the appearance of the notion of quasi-inertial frames as a drawback but rather as an advantage. Such a theory of gravitation is more concrete, more substantialistic than the original one.

Since the gravitational field is determined uniquely by the non-linear equation for the universal spinor field, the traditional equation for the gravitational field becomes spurious. This « equation » is to be understood simply as a definition of the energy-momentum-stress tensor of the fundamental spinor field. The field ψ, χ appears to be inseparably connected with its own gravitation so that it makes no sense to look for another definition of T_{kl} except for

$$R_{kl} - \tfrac{1}{2} g_{kl} R \underset{\text{def}}{=} \varkappa T_{kl}\,, \tag{13}$$

where $\varkappa$ is a factor of proportionality. Up to this factor T_{kl} is nothing else but the left hand side of (13) similarly as in the special relativity the mass is nothing else but the energy (up to the factor c^2). In this way the old problem: what exactly is to be put to the right hand side of an « equation » (13) acquires an unexpectedly banal solution.

Another advantage of the present formulation is as follows: we cannot change the sign of f^{kl} from case to case but have to decide it once for ever. This settles principally the question whether (or under which distribution of matter described by the universal spinor field) antigravitation is possible. In the traditional relativity the sign (and magnitude) of the gravitational coupling constant was quite arbitrary. There existed solutions of the type of a repulsion for situations where this is surely not the case. Now, the sign is settled once for ever and the magnitude of the coupling constant seems to be related to (the sixth power of) the Fermi coupling constant.

Concluding let us state that it seems possible not only to include gravitation as an intimate and inseparable property of the universal spinor field of Heisenberg but, at the same time, to throw a new light upon the theory of gravitation.

([4]) L. Infeld: *Rev. Mod. Phys.*, **29**, 3, 398 (1957).

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Hyperon Polarization in Associated Production and Heavy Meson Parity.

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(ricevuto il 26 Maggio 1958)

Recently a number of experiments involving strange particles have been proposed (1) in order to determine from experimental data the H-K relative parity. It is the purpose of this note to suggest accurate measurements of the angular dependence of the polarization of the hyperons, produced in associated production by pions, as a rather sensitive test of the hypotheses concerning the H-K parity. We shall show that quantitative estimates of the polarization intensity can be obtained from the differential cross sections, once the K-H-N interaction is treated as sensibly weaker than the π-N interaction (2).

It is clear that, due to the associated production of K and H required by strangeness conservation and non-conservation of parity in decay processes, only the parity of the K-meson relative to the hyperons can be defined. We shall however in the following use the expressions «scalar» and «pseudoscalar» K-meson, defining therefore the parity of the hyperon to be equal to that of the nucleons.

In what follows two basic assumptions are made: the spins of the strange particles are supposed to be $\frac{1}{2}$ for the hyperons (3) and 0 for the K-meson (4); only S and P waves are admitted in the final H-K states.

(1) R. H. Dalitz: *Proceedings of the Sixth Annual Rochester-Conference* (1956); D. Amati and B. Vitale: *Nuovo Cimento*, **6**, 395 (1957); G. Costa and B. T. Feld: *Phys. Rev.*, **109**, 606 (1958).

(2) The possibility of measuring the hyperon polarization in decay processes is analysed by: T. D. Lee, J. Steinberger, G. Feinberg, P. K. Kabir and C. N. Yang: *Phys. Rev.*, **106**, 1367 (1957).

(3) See, for instance, F. Eisler, R. Plano, A. Prodell, N. Samios, M. Schwartz, J. Steinberger, P. Bassi, V. Borelli, G. Puppi, G. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, I. Manelli, R. Santangelo, V. Silvestrini, G. L. Brown, D. A. Glaser and C. Graves: *Nuovo Cimento*, **7**, 222 (1958).

(4) See, for instance, M. Baldo-Ceolin, A. Bonetti, W. D. B. Greening, S. Limentani, M. Merlin and G. Vanderhaeghe: *Nuovo Cimento*, **6**, 84 (1957); report of R. Karplus: *Conference on Elementary Particles* (Venice 1957); R. H. Dalitz: *Phys. Rev.*, **99**, 915 (1955).

We shall write down the most general expression for the transition matrix in associated production processes:

$$\pi + \mathrm{N} = \mathrm{H} + \mathrm{K}\,, \tag{1}$$

where H stands for a Λ^0 or a Σ hyperon. The possible transitions are:

$$\begin{array}{ll} \text{Scalar } K & \text{Pseudoscalar } K \\ \left\{\begin{array}{l} S_{\frac{1}{2}} \to P_{\frac{1}{2}} \\ P_{\frac{1}{2}} \to S_{\frac{1}{2}} \\ D_{\frac{3}{2}} \to P_{\frac{3}{2}} \end{array}\right. & \begin{array}{l} S_{\frac{1}{2}} \to S_{\frac{1}{2}} \\ P_{\frac{1}{2}} \to P_{\frac{1}{2}} \\ P_{\frac{3}{2}} \to P_{\frac{3}{2}} \end{array} \end{array} \tag{2}$$

The transition matrix is then expressed by (5),

$$M_s = a(\boldsymbol{p}_{\mathrm{H}}\cdot\boldsymbol{\sigma}_{\mathrm{H}}) + b(\boldsymbol{p}_{\pi}\cdot\boldsymbol{\sigma}_{\mathrm{H}}) + c(\boldsymbol{p}_{\mathrm{H}}\cdot\boldsymbol{p}_{\pi})(\boldsymbol{p}_{\pi}\cdot\boldsymbol{\sigma}_{\mathrm{H}})\,, \tag{3}$$

$$M_{ps} = A + B(\boldsymbol{p}_{\pi}\cdot\boldsymbol{p}_{\mathrm{H}}) + C(\boldsymbol{p}_{\mathrm{H}}\times\boldsymbol{p}_{\pi}\cdot\boldsymbol{\sigma}_{\mathrm{H}})\,, \tag{4}$$

where s and ps stand respectively for « scalar » and « pseudoscalar » K; $\boldsymbol{p}_{\mathrm{H}}$ and $\boldsymbol{p}_{\pi}$ are the hyperon and pion momenta in the centre of mass and $\boldsymbol{\sigma}_{\mathrm{H}}$ is the hyperon spin operator.

From (3) and (4) we easily get the following expressions for the differential cross section and the polarization vector:

$$\sigma_s(\vartheta) = |a|^2 p_{\mathrm{H}}^2 + |b|^2 p_{\pi}^2 + 2(\mathrm{Re}\, a^+b + \mathrm{Re}\, b^+cp_{\pi}^2)p_{\mathrm{H}}p_{\pi}\cos\vartheta + \\ + (|c|^2 + 2\ \mathrm{Re}\, a^+c)p_{\pi}^2 p_{\mathrm{H}}^2 \cos^2\vartheta\,, \tag{5}$$

$$\boldsymbol{P}_s(\vartheta) = (\mathrm{Im}\, a^+b\, p_{\mathrm{H}}p_{\pi}\sin\vartheta + \mathrm{Im}\, a^+c\, p_{\mathrm{H}}^2 p_{\pi}^2 \sin 2\vartheta)\boldsymbol{n}/\sigma_s(\vartheta)\,, \tag{6}$$

$$\sigma_{ps}(\vartheta) = |A|^2 + |C|^2 p_{\pi}^2 p_{\mathrm{H}}^2 + 2\ \mathrm{Re}\, A^+B p_{\mathrm{H}}p_{\pi}\cos\vartheta + (|B|^2 - |C|^2)p_{\pi}^2 p_{\mathrm{H}}^2\cos^2\vartheta\,, \tag{7}$$

$$\boldsymbol{P}_{ps}(\vartheta) = (2\ \mathrm{Re}\, A^+C p_{\mathrm{H}}p_{\pi}\sin\vartheta + \mathrm{Re}\, B^+C p_{\mathrm{H}}^2 p_{\pi}^2 \sin 2\vartheta)\boldsymbol{n}/\sigma_{ps}(\vartheta)\,, \tag{8}$$

where $\boldsymbol{n}$ is a unit vector normal to the $\boldsymbol{p}_{\pi}\boldsymbol{p}_{\mathrm{H}}$ plane.

The estimate of the coefficients present in (3) and (4) could in principle give a useful tool for determining the K parity. However, because of the high energy in the initial state and therefore of the many final channels open, it would be exceedingly complicated to obtain a quantitative estimate of the coefficients from experiment.

If the analysis of the π-N interaction at energy around 800 MeV could be expressed in terms of phase shifts, a very naive approach could be tried. One could follow lines already introduced by WATSON (6) who was able to express the pion photoproduction amplitudes in terms of the pion-nucleon scattering phase shifts. This

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(5) Very general expressions for the polarization of particles with spin $\frac{1}{2}$ have been recently given by R. SPITZER and H. P. STAPP: *Phys. Rev.*, **109**, 540 (1958).

(6) K. WATSON: *Phys. Rev.*, **95**, 228 (1954).

means that we shall consider the K-H-N interaction as rather weaker than the π-N interaction and therefore disregard the final state interaction between K and H. (We note that the possibility that the K-H-N interaction is weaker than the π-N interaction has been discussed in detail by GELL-MANN on the basis of the present experimental evidence) [7].

We shall treat first the scalar case. Let us consider, for instance, the following process:

$$\pi^- + p = \Lambda^0 + K^0 . \tag{9}$$

The final state is a pure $T=\frac{1}{2}$ state and, if the interaction is supposed to be charge independent the transition takes place only through the $T=\frac{1}{2}$ channel. The coefficients appearing in (3) can therefore be defined by:

$$a = ia_S \exp[i\delta_1]\,; \qquad b = ib_P \exp[i\delta_{13}]\,; \qquad c = ic_D \exp[i\delta^D_{13}]\,, \tag{10}$$

where the δ are the pion-nucleon scattering phase shifts and a_S, b_P and c_D are now real quantities.

As soon as the phase shifts appearing in (10) will be known with not too large an error at the production energies (around 800 MeV in the laboratory system) we shall be able to get the three real coefficients from the observed differential cross-section (by using (5)) and to use them in order to calculate the expected polarization intensity (by using (6)). We easily obtain indeed:

$$\begin{aligned}\sigma_s^\Lambda(\vartheta) = a_S^2 p_H^2 + b_P^2 p^2 + 2\big(a_S b_P \cos(\delta_{13} - \delta_1) + b_P c_D \cos(\delta^D_{13} - \delta_{13}) p_\pi^2\big) p_H p_\pi \cos\vartheta\, + \\ + \big(c_D^2 + 2a_S c_D \cos(\delta^D_{13} - \delta_1)\big) p_\pi^2 p_H^2 \cos^2\vartheta\,,\end{aligned} \tag{11}$$

$$P_s(\vartheta) = \big(a_S b_P \sin(\delta_{13} - \delta_1)\sin\vartheta + a_S c_D \sin(\delta^D_{13} - \delta_1) p_H p_\pi \sin 2\vartheta\big) p_H p_\pi / \sigma(\vartheta)\,. \tag{12}$$

In the pseudoscalar case $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ phase shifts will both contribute to B and C; the possibility of analysing the data from the point of view here proposed is then much less reliable.

A similar analysis, although more complicated because of the two isotopic channels open, can also be performed for the process:

$$\pi + N \to \Sigma + H\,. \tag{13}$$

[7] M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

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Angular Distribution of Photoprotons from Silicon.

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(ricevuto il 31 Maggio 1958)

Natural Silicon is a mixture of the isotopes ^{28}Si (92.2%), ^{29}Si (4.7%) and ^{30}Si (3.1%); threshold energy for the $Si(\gamma, p)$ processes are respectively 11.6, 12.3 and 12.9 MeV (1). The (γ, p) reactions in the less abundant isotopes ^{29}Si and ^{30}Si lead to ^{28}Al, 2.3 min half-life, and to ^{29}Al, 6.6 min half-life. The activation curves resulting from these reactions have been studied with a bremsstrahlung beam by the Canadian group (2) who found cross sections peaked around 21 MeV.

The $^{29}Si(\gamma, p)$ and $^{30}Si(\gamma, p)$ cross sections relative to $Cu(\gamma, n)$ cross section have also been determined by Hirzel and Wäffler (3) by means of activation with the monocromatic γ-rays from the $^{7}Li(p, \gamma)$ reaction.

We have studied the photoprotons from natural silicon irradiated with a collimated 30 MeV bremsstrahlung beam of the B.B. betatron of Turin.

The experimental technique is described in a previous work (4).

The Si target (a disk 4 mm diameter, 0.75 mm thick) was placed in an evacuated exposure chamber together with the 200 μm Ilford C2 nuclear emulsions which detected the photoprotons.

The nuclear emulsion was parallel to the γ-rays. The scanned area of emulsion was hit by photoprotons entering at a dip angle β in the interval 15° to 20° and ejected from the Si target with angle ϑ in the interval 16° to 164° with respect to the γ-ray beam.

From the geometrical arrangement of the Si target with respect to the nuclear emulsion it was easy to test if the observed photoprotons came from the target. The background is about 15% and is due mainly to the low energy protons (below 3.5 MeV).

The angular distribution obtained by grouping the protons into angular intervals corresponding to equal solid angles is reported in Fig. 1 for protons with energy $E'_p > 7$ MeV, curve (*a*), $E'_p > 5.5$ MeV, curve (*b*) and for $E'_p < 3.5$ MeV. E'_p is

(1) C. W. Li: *Phys. Rev.*, **88**, 1038 (1952).

(2) L. Katz, R. N. H. Haslam, J. Goldemberg and J. G. V. Taylor: *Can. Journ. Phys.*, **32**, 580 (1954).

(3) O. Hirzel and H. Wäffler: *Helv. Phys. Acta*, **20**, 370 (1947).

(4) C. Milone, R. Ricamo and A. Rubbino: *Nuovo Cimento*, **5**, 528 (1957).

the energy lost by the protons inside the emulsion. Obviously, E'_p is not greater than the ejection energy E_p in the (γ, p) process. The angular distributions (a) and (b) are anisotropic and show a forward asymmetry peaked around 60°. The experimental points can be fitted by functions of the type $1+(B \text{ sen } \vartheta + C \sin\vartheta \cos\vartheta)^2$, with $B=0.8$ and $C=0.6$ for the curve (a) and $B=0.7$ and $C=0.4$ for the curve (b). This corresponds to E1 dipole absorption with superposed contribution of higher multipole terms.

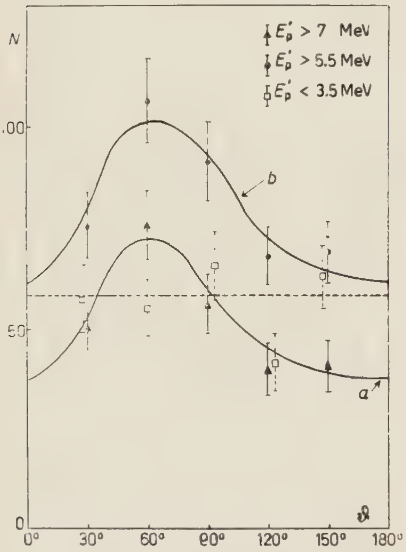

Fig. 1. – Angular distribution of photoprotons from Si. N = number of photoprotons observed with energy E_p for constant solid angle, and 30° ϑ-interval (arbitrary units). Standard deviations are included.

The proton group with $E'_p < 3.5$ MeV certainly include a number of tracks generated with rather high E_p: however their angular distribution is practically isotropic, inside the experimental errors.

The Si target was thick (0.75 mm/$/\sin\beta \sim 2.5$ mm) so that the range differential spectrum could be obtained by derivation of the experimental curve. The energy spectrum was then easily derived.

The derivation of a curve affected by appreciable statistical errors, gives results having a large inherent uncertainty. Besides, the high energy tail of the spectrum is not at all known because of the tracks going out of our emulsion layer. Therefore we confine ourselves to showing a qualitative — but well defined — result of this procedure, that is, that the energy spectrum has a maximum around 6 MeV, in agreement with the other known photoproton spectra.

We can conclude that in the Si photoproton spectrum there is a rather strong anisotropic component due to its direct emission in the energy range $E_p > 5$ MeV, while the isotropy of the $E'_p < 3.5$ MeV component indicates that probably a major part of it is emitted after the formation of a compound nucleus.

In a previous work [4] we have studied with the same experimental technique the photoprotons from C. We may compare the photoprotons' yield from Si to that of C.

Assuming the spectra from Si and C not to be very different (the maxima are found for about the same energy) and taking into account the thickness of the target that contributes to give photoprotons in the plates, we roughly obtain a yield ratio Si/C ~ 3.7. An approximate integrated cross section of about 0.23 MeV-barn may be derived for the natural Si taking account of the C integrated cross section [5,6]. This figure agrees with the figure of 0.27 MeV-barn obtained by JOHANSSON [7]. The values found by the Canadian group [4]

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(5) J. HALPERN and A. K. MANN: *Phys. Rev.*, **83**, 370 (1951).

(6) L. COHEN, A. K. MANN, B. J. PATTON, K. REIBEL, W. E. STEPHENS and E. J. WINHOLD: *Phys. Rev.*, **104**, 108 (1956).

(7) S. A. E. JOHANSSON: *Phys. Rev.*, **97**, 1186 (1955).

for the less frequent isotopes ^{29}Si and ^{30}Si are respectively 0.26 and 0.19 MeV-barn.

The Canadian group [4] has found for $^{28}Si(\gamma, n)$ an integrated cross section 0.075 MeV-barn. Indeed we may derive for both reactions (γ, p) and (γ, n) in ^{28}Si an integrated cross section of about 0.30 MeV-barn. The figure that may be derived for the « total » cross section according to the expression:

$$\int_0^\infty \sigma_c(\gamma)\,\mathrm{d}E = 0.015\,A(1+0.8x) \qquad \text{for } x=0 \;\; [8,9],$$

is 0.42 MeV-barn.

* * *

Our thanks are due to Prof. R. Ricamo for helpful suggestions and stimulating discussions and to Prof. G. Cortini for useful discussion.

(8) J. S. Levinger and H. A. Bethe: *Phys. Rev.*, **78**, 115 (1950).
(9) R. Montalbetti, L. Katz and J. Goldemberg: *Phys. Rev.*, **91**, 659 (1953).

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Remarks on Some Identities in Static Meson Theory.

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Chew and Low (1), and Wick (2) demonstrate the following identities:

$$(H - E_0 - \omega_q)\, a^\dagger_{q,\alpha} \Psi^0 = V^{(0)}_{q,\alpha} \Psi^0 , \tag{1}$$

and

$$(H - E_0 + \omega_q)\, a_{q,\alpha} \Psi^0 = - V^{(0)\dagger}_{q,\alpha} \Psi^0 . \tag{2}$$

The equation

$$a_{q,\alpha} \Psi^0 = -\left(\frac{1}{H - E_0 + \omega_q}\right) V^{(0)\dagger}_{q,\alpha} \Psi^0 \tag{3}$$

can be obtained by dividing eq. (2) by $(H - E_0 + \omega_q)$. (3) is an important identity which is repeatedly used in static meson theory (1,2). However, the expression obtained by dividing (1) by $(H - E_0 - \omega_q)$,

$$a^\dagger_{q,\alpha} \Psi^0 = \left(\frac{1}{H - E_0 - \omega_q}\right) V^{(0)}_{q,\alpha} \Psi^0 \tag{4}$$

is wrong. The reason for the failure of (4) is clearly related to the fact that the operator $(H - E_0 - \omega_q)$ has 0 as one of its eigenvalues, since the simple proof of (3) fails in this case (2). However, the prohibition against dividing by operators that

(*) This research was supported by the U.S.A.F. through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. 18(603)-108.

(1) G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1570 (1956).

(2) G. C. Wick: *Rev. Mod. Phys.*, **27**, 339 (1955).

give rise to the eigenvalue 0 is not general. Division by a non-positive (or negative)-definite operator is, in fact, a part of the usual derivation of the scattering equation [1], and the equation

$$\chi^{(\pm)}_{q\,\alpha} = -\left(\frac{1}{H-E_0-\omega_q \mp i\eta}\right) V^{(0)}_{q,\alpha}\Psi^0 \tag{5}$$

is commonly obtained in this fashion.

That (4) is incorrect is easily established by considering the case of neutral scalar theory. In this case, $V^{(0)}_q$ and H commute and (4) leads to

$$a^{\dagger}_q\Psi^0 = -(V^{(0)}_q/\omega_q)\Psi^0\,, \tag{6}$$

whereas (3) leads to

$$a_q\Psi^0 = -(V^{(0)\dagger}_q/\omega_q)\Psi^0\,. \tag{7}$$

By explicit construction from the exact solution of the neutral scalar theory, it can be seen that (6) is wrong but that (7) is correct. Since Ψ^0 has a non-zero eigenvalue of the operator $(H-E_q)$, $\big(1/(H-E_q)\big)V^{(0)}_q\Psi^0$, in the neutral scalar theory, is a well-defined unambiguous function, and the failure of eq. (4), in this case, cannot be ascribed to an ambiguity in the right hand side of the equation. Since eqs. (3) and (5) are correct, but (4) is not, it becomes of interest to establish a criterion for deciding when it is permissible to divide an equation by an operator which has 0 as one of its eigenvalues.

Let us consider an operator Ω_a and its complete set of eigenfunctions, denoted by r_k. Then $\Omega_a r_k = \bar{\Omega}_a(k) r_k$ and $(1/\Omega_a)r_k = [1/\bar{\Omega}_a(k)]r_k$. Let us assume that the spectrum of eigenvalues, $\bar{\Omega}_a(k)$, is continuous between a value corresponding to $k=\varkappa$ and ∞, and that $\bar{\Omega}_a(k)$ is a continuous monotonically increasing function of k. Let us furthermore assume that $\bar{\Omega}_a(a)=0$ and that $a>\varkappa$. We will consider the equation $\Omega_a f = g$ (where f and g are represented as $f=\int_\varkappa^\infty \mathrm{d}k\,F(k)r_k$ and $g=\int_\varkappa^\infty \mathrm{d}k\,G(k)r_k$) and examine the condition under which $(1/\Omega_a)g=f$. We will consider the following kinds of functions: A) $F(k)$ is bounded for $k>\varkappa$. B) $F(k)$ can be represented as $F(k)=\mathcal{F}(k)+\lambda(k)\,\delta(k-a)$ where $\mathcal{F}(k)$ is bounded for $k>\varkappa$. C) $F(k)$ is of the form $\varphi(k)/(k-a)$, where $\varphi(k)$ is bounded for $k>\varkappa$ and continuous at $k=a$. If f is written in the form

$$f=\int_\varkappa^\infty \mathrm{d}k\left[X(k)\frac{\mathcal{P}}{k-a}+Y(k)\,\delta(k-a)\right]r_k$$

(where $\mathcal{P}$ denotes the principal value part and where $X(k)$ and $Y(k)$ are bounded functions), then it is sufficiently general to correspond to type A), B), C) functions. For case C), $X(k)=\varphi(k)$, and $Y(k)=\varphi(k)\gamma$ ($\gamma=0,\ \pm i\pi$). For case B), $X(k)=\mathcal{F}(k)[k-a]$ and $Y(k)=\lambda(a)$. For case A), $X(k)=F(k)[k-a]$ and $Y(k)=0$. When Ω_a is applied to f we get

$$g=\int_\varkappa^\infty \mathrm{d}k\,\bar{\Omega}_a(k)X(k)\frac{\mathcal{P}}{k-a}\,r_k\,,$$

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since $\bar{\Omega}_a(a)\,\delta(k-a)=0$, and

$$\frac{1}{\Omega_a}\,g=\int\limits_{\varkappa}^{\infty}\mathrm{d}k\,X(k)\frac{\mathcal{P}}{k-a}\,r_k\,.$$

Hence, for case A), the identity $(1/\Omega_a)\,\Omega_a f=f$ is proved; however, in cases B) and C), the identity does not hold since the $Y(a)r_a$ part of f does not reappear in $(1/\Omega_a)g$.

Let us define an operator $(1/\Omega_a)_{[r\,f]}$, (with explicit dependence on r and f) so that

$$\left(\frac{1}{\Omega_a}\right)_{[r\,f]}\Omega_a f=\left(\frac{1}{\Omega_a}\right)\Omega_a f+Y(a)\,r_a\,.$$

This operator, then, is the inverse of Ω_a for functions of type A), B), C). In the case of type C) functions, the operator $(1/\Omega_a)_{[r.f]}$ differs from $(1/\Omega_a)$ only by including the prescription to choose the contour in evaluating $(1/\Omega_a)g$ in the same way as in expanding the f in terms of the r_k. It is application of this operator rather than just division by Ω_a which is properly made use of in deriving (5). This can also easily by seen by realizing that since

$$(H-E_q)\,\chi^{(\pm)}_{q\,\alpha}=-V^{(0)}_{q.\alpha}\Psi^0\,,$$

division by $(H-E_q)$ would lead to the erroneous conclusion that $\chi^{(+)}_{q.\alpha}=\chi^{(-)}_{q.\alpha}$. If $(1/\Omega_a)_{[r.f]}$ is applied to (1),

$$a^{\dagger}_{q,\alpha}\Psi^0=\left(\frac{1}{H-E_q}\right)_{[\,\bar{\Psi}^{(+)}_{k.\beta};\,a^{\dagger}_{q,\alpha}\Psi^0]}V^{(0)}_{q.\alpha}\Psi^0$$

is obtained. ($\bar{\Psi}^{(+)}_{k.\beta}$ defines the complete set consisting of the ground state Ψ^0 and all the outgoing waves $\Psi^{(+)}_{k_1;\beta_1}$, $\Psi^{(+)}_{k_1.k_2;\,\beta_1.\beta_2}$, $\Psi^{(+)}_{k_1\dots k_n;\,\beta_1\dots\beta_n}$; it will be assumed that no stable isobars exist.) The quantity $\langle\Psi^{(+)}_{k,\beta}|a^{\dagger}_{q,\alpha}\Psi^0\rangle$ is given by

$$\begin{aligned}\langle\Psi^{(+)}_{k.\beta}|a^{\dagger}_{q,\alpha}\Psi^0\rangle&=\delta_{k\,q}\delta_{\alpha.\beta}+\langle\Psi^0|V^{(0)}_{q,\alpha}\left(\frac{1}{H-E_0+\omega_q}\right)\left(\frac{1}{H-E_0+\omega_k}\right)V^{(0)\dagger}_{k,\beta}|\Psi^0\rangle+\\&+\frac{1}{\omega_k-\omega_q-i\eta}\left\{\langle\Psi^0|V^{(0)}_{q.\alpha}\left(\frac{1}{H-E_0+\omega_q}\right)V^{(0)\dagger}_{k.\beta}|\Psi^0\rangle+\right.\\&\left.+\langle\Psi^0|V^{(0)\dagger}_{k,\beta}\left(\frac{1}{E_k-H-i\eta}\right)V^{(0)\dagger}_{q.\alpha}|\Psi^0\rangle\right\}\end{aligned}$$

and, in applying $1/(H-E_q)$ to $a^{\dagger}_{q,\alpha}\Psi^0$ when this latter quantity is expanded in the complete set specified, the terms proportional to $\delta(\omega_q-\omega_k)$ are a) the $\delta_{\alpha,\beta}\delta_{k\,q}$ from commuting $a^{\dagger}_{q,\alpha}$ and $a_{k.\beta}$, and b) the contribution from $(\omega_k-\omega_q-i\eta)^{-1}$. The application of $\left(1/(H-E_q)\right)_{[\bar{\Psi}^{(+)}_{k,\beta};\,a^{\dagger}_{q,\alpha}\Psi^0]}$ thus amounts to choosing the contour $1/(H-E_q-i\eta)$ and adding an additional term $\delta_{k,q}\delta_{\alpha.\beta}\Psi^{(+)}_{k.\beta}$. The only $\delta(\omega_k-\omega_q)$ terms from the two or more meson wave functions are terms like $(\omega_{k_1}+\omega_{k_2}+\dots+$

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$+\omega_{k_n}-\omega_q-i\eta)^{-1}$ and these are automatically handled correctly due to the choice of contour made. For example, in the case of the two-meson wave function,

$$\Psi^{(+)}_{k_1,k_2;\beta_1,\beta_2}=a^\dagger_{k_1,\beta_1}a^\dagger_{k_2,\beta_2}\Psi^0-\left(\frac{1}{H-E_0-\omega_{k_1}-\omega_{k_2}-i\eta}\right)[V^{(0)}_{k_1,\beta_1}a^\dagger_{k_2,\beta_2}+V^{(0)}_{k_2,\beta_2}a^\dagger_{k_1,\beta_1}]\Psi^0$$

and $\langle\Psi^{(+)}_{k_1,k_2;\beta_1,\beta_2}|a^\dagger_{q,\alpha}\Psi^0\rangle$ is given by:

$$\begin{aligned}\langle\Psi^{(+)}_{k_1,k_2;\beta_1,\beta_2}|a^\dagger_{q,\alpha}\Psi^0\rangle=-\Bigg\|&\langle\Psi^0|V^{(0)}_{q,\alpha}\left(\frac{1}{H-E_0+\omega_q}\right)\left(\frac{1}{H-E_0+\omega_{k_1}+\omega_{k_2}}\right)\cdot\\
&\cdot\left[V^{(0)\dagger}_{k_2,\beta_2}\left(\frac{1}{H-E_0+\omega_{k_1}}\right)V^{(0)\dagger}_{k_1,\beta_1}+V^{(0)\dagger}_{k_1,\beta_1}\left(\frac{1}{H-E_0+\omega_{k_2}}\right)V^{(0)\dagger}_{k_2,\beta_2}\right]|\Psi^0\rangle+\\
&+\frac{1}{\omega_{k_1}+\omega_{k_2}-\omega_q-i\eta}\Bigg\{\langle\Psi^0|\Bigg\{V^{(0)}_{q,\alpha}\left(\frac{1}{H-E_0+\omega_q}\right)\left[V^{(0)\dagger}_{k_1,\beta_1}\left(\frac{1}{H-E_0+\omega_{k_2}}\right)V^{(0)\dagger}_{k_2,\beta_2}+\right.\\
&+\left.V^{(0)\dagger}_{k_2,\beta_2}\left(\frac{1}{H-E_0+\omega_{k_1}}\right)V^{(0)\dagger}_{k_1,\beta_1}\right]+V^{(0)\dagger}_{k_1,\beta_1}\left(\frac{1}{E_{k_1}-H-i\eta}\right)V^{(0)}_{q,\alpha}\left(\frac{1}{H-E_0+\omega_{k_2}}\right)V^{(0)\dagger}_{k_2,\beta_2}+\\
&+V^{(0)\dagger}_{k_2,\beta_2}\left(\frac{1}{E_{k_2}-H-i\eta}\right)V^{(0)}_{q,\alpha}\left(\frac{1}{H-E_0+\omega_{k_1}}\right)V^{(0)\dagger}_{k_1,\beta_1}-\left[V^{(0)\dagger}_{k_1,\beta_1}\left(\frac{1}{E_{k_1}-H-i\eta}\right)V^{(0)\dagger}_{k_2,\beta_2}+\right.\\
&+\left.V^{(0)\dagger}_{k_2,\beta_2}\left(\frac{1}{E_{k_2}-H-i\eta}\right)V^{(0)\dagger}_{k_1,\beta_1}\right]\left(\frac{1}{E_0+\omega_{k_1}+\omega_{k_2}-H-i\eta}\right)V^{(0)}_{q,\alpha}\Bigg\}|\Psi^0\rangle\Bigg\}\Bigg\|\,.\end{aligned}$$

As pointed out above, the only $\delta(\omega_{k_1}+\omega_{k_2}-\omega_q)$ contribution comes from $(\omega_{k_1}+\omega_{k_2}-\omega_q-i\eta)^{-1}$.

From this, $a^\dagger_{q,\alpha}\Psi^0=\Psi^{(+)}_{q,\alpha}+\big(1/(H-E_q-i\eta)\big)V^{(0)}_{q,\alpha}\Psi^0$ is correctly obtained. Similarly, if $a^\dagger_{q,\alpha}\Psi^0$ is expanded in terms of the complete set in which the $\overline{\Psi}^{(-)}_{k,\beta}$ are included instead of the $\overline{\Psi}^{(+)}_{k,\beta}$,

$$a^\dagger_{q,\alpha}\Psi^0=\Psi^{(-)}_{q,\alpha}+\frac{1}{H-E_q+i\eta}V^{(0)}_{q,\alpha}\Psi^0$$

is correctly obtained. In the case of neutral scalar theory, these expressions correctly reduce to the exact solution for $a^\dagger_q\Psi^0$.

* * *

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Phenomenological Relation between Electron and Photon Disintegration of Nuclei.

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(ricevuto il 6 Giugno 1958)

It is well known that phenomena produced by photons and by electrons are intimately connected.

It has been shown that in electric dipole approximation the electron [2] disintegration total cross-section is equal to the total photon disintegration cross-section multiplied by [3]

$$\int N(\boldsymbol{p}_i, \boldsymbol{p}_f)\, \mathrm{d}\Omega_{p_f}\,,$$

were $N(p_i, p_f)$ represents the number photons contained in the electromagnetic field created by the electron transition from the state of momentum p_i to a state of momentum p_f and is given by

$$N = \frac{e^2}{2m^2 k_0^2}\left[-\frac{1}{2}-\frac{E_i^2+E_f^2}{k_0^2-k^2}-\frac{2m^2k_0^2}{(k_0^2-k^2)^2}\right] \tag{1}$$

$$k_0 = E_i - E_f$$

$$\boldsymbol{k} = \boldsymbol{p}_i - \boldsymbol{p}_f\,.$$

We want here to show that a similar relation exists (in electric dipole approximation) between all electron and photon differential cross-sections.

For sake of definiteness we shall fix our attention on the case of the disintegration of deuteron; it will become clear that our proof applies to any two body disintegration of nuclei.

The matrix element for an electromagnetic transition induced by an electron is given by [1]:

$$H = \sum_\mu \langle \alpha | J_\mu | \alpha' \rangle A_\mu\,, \tag{2}$$

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[1] As usual greek indexes refer to four dimensional summation, latin indexes to three dimensional one.

[2] E. Guth and C. J. Mullin: *Phys. Rev.*, **76**, 234 (1949).

[3] Apart some factor in which we are not interested here.

where J_μ is the current between the states α and α' representing the initial and final states of the nuclear system,

$$A_\mu = \frac{\bar{u}(p-k)\gamma_\mu u(p)}{k_0^2 - k^2} \tag{3}$$

is the field produced by the electron. Because of the conservation laws of the nuclear and electron charge (4)

$$K_\mu \langle \alpha | J_\mu | \alpha' \rangle = 0\,,$$

$$K_\mu A_\mu = 0\,.$$

Then Eq. (2) can be rewritten as:

$$H = \langle \alpha | \boldsymbol{J} | \overline{\alpha}' \rangle \cdot \boldsymbol{A}'\,, \tag{4}$$

where

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{k}\frac{(\boldsymbol{k}\cdot\boldsymbol{A})}{k_0^2}\,.$$

In electric dipole approximation the matrix element $\langle \alpha | \boldsymbol{J} | \alpha' \rangle$ does not depend on the momentum of the incoming photon but on the relative momentum $\boldsymbol{p}$ of the ejected nucleons and on the initial and final state variables.

In order to compute the differential cross-sections one has to take the square of H and to sum over all final nucleon and electron spin states and to average over the initial state variables.

The differential cross-section will therefore be given by:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega_e\,\mathrm{d}\Omega_N} = \frac{2\pi}{v_e} J_{ij} N_{ij}\, \varrho_e \varrho_N\,. \tag{5}$$

In this formula v_e is the velocity of incoming electron ϱ_e and ϱ_N are the final states density for the electron and nucleon respectively and

$$J_{ij} = \left(\langle \alpha | J_i | \alpha' \rangle^* \langle \alpha | J_i | \alpha' \rangle\right)\,,$$

$$N_{ij} = S(A_i'^* A_j)\,,$$

where S means proper sum and average.

The explicit expression for N_{ij} is:

$$N_{ij} = \frac{e^2}{2m^2(k_0^2 - k^2)}\left\{2p_i p_j + \frac{1}{2}(k^2 - k_0^2)\,\delta_{ij} + (p_i k_j + p_j k_i + k_i k_j)\cdot \right.$$
$$\left. \cdot\left(-1 + \frac{k^2}{k_0^2} - \frac{2\boldsymbol{p}\cdot\boldsymbol{k}}{k_0^2}\right) + \frac{k_i k_j}{k_0^4}\left[2(\boldsymbol{p}\cdot\boldsymbol{k})^2 - 2k^2\boldsymbol{p}\cdot\boldsymbol{k} + \frac{1}{2}(k^2 - k_0^2)k^2\right]\right\}. \tag{6}$$

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(4) See for example: R. H. Dalitz and D. R. Yennie: *Phys. Rev.*, **105**, 1598 (1957).

Because of our electric dipole approximation J_{ij} will only depend on the vector q_j and therefore its form will be

$$J_{ij} = \alpha(E)\,\delta_{ij} + \beta(E)\frac{q_i q_j}{q^2}\,. \tag{7}$$

Therefore the differential electron disintegration cross section is given by:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\Omega_e\,\mathrm{d}\Omega_N} = \frac{2\pi}{v_e}\,\varrho_e\,\varrho_N\left\{N\alpha(E) + \frac{\beta(E)}{q^2}\,N_{ij}q_i q_j\right\}, \tag{8}$$

where $N = \sum_i N_{ii}$ is given in Eq. (1).

Eq. (8) gives all differential cross sections in terms of two functions $\alpha(E)$ and $\beta(E)$. It is simple to show that such functions are directly connected with the photo disintegration cross-section.

Indeed such a cross-section is given by:

$$\frac{\mathrm{d}\sigma_{ph}}{\mathrm{d}\Omega_N} = \frac{2\pi}{c}\,\varrho_N J_{ij}\,\varepsilon_i\,\varepsilon_j = \frac{2\pi}{c}\,\varrho_N\left[\alpha(E) + \frac{\beta(E)}{q^2}\,(\boldsymbol{q}\cdot\boldsymbol{\epsilon})^2\right], \tag{9}$$

where $\boldsymbol{\epsilon}$ is the photon polarization.

Average on the photon polarization gives

$$\frac{\mathrm{d}\sigma_{\mathrm{ph}}}{\mathrm{d}\Omega_N} = \frac{2\pi}{c}\,\varrho_N[\alpha(E) + \beta(E)\sin^2\theta)]\,, \tag{10}$$

θ is the angle between the incoming photon and $\boldsymbol{q}$.

We conclude that until electric dipole approximation is valid electron induced processes do not give any new information about the nuclear matrix element. This conclusion of course applies also to processes induced by polarized photons.

We want to point out that such a simple relation is only limited to electric dipole; an experimental check of such relation could give the limits of validity of the electric dipole approximation.

* * *

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Pion Production in the K$^+$-Nucleon Interaction.

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(ricevuto il 16 Giugno 1958)

Recently experimental work concerning the K$^+$-nucleon scattering has reached an energy range (between 220 up to 350 MeV) (1) where secondary pion production becomes energetically possible; in fact a few events of this kind have so far been reported (2). It may therefore be of some interest to try to calculate the value and the behaviour of the cross-section for this effect.

The point of more general interest in the study of this problem should consist perhaps in examining whether pion production in K$^+$-nucleon interactions should turn out to be more or less sensitively dependent on the type of interaction according to which the K-mesons are supposed to react with nucleons. In fact, up to now, although several works have been done on different phenomena involving K$^+$-nucleon interactions (3), no definite ideas have been obtained on this point.

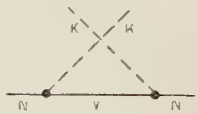

Fig. 1.

Generally, K$^+$-nucleon scattering has been interpreted according to the scheme of Fig. 1, and the best agreement between theory and experimental data appears in this case to be obtained when the K$^+$-N-Y interactions in the two vertices of the scheme are assumed to be pseudoscalar.

However, this pseudoscalar character of the K$^+$-N-Y interaction cannot be considered as being up to now confirmed in other phenomena where this interaction

(1) B. SECHI-ZORN and G. T. ZORN: *Phys. Rev.*, **108**, 1098 (1957); D. KEEFE, A. KERNAN and A. MONTWILL (priv. circulated preprint); E. HELMY, J. H. MULVEY, D. J. PROWSE, D. M. STORK, M. GRILLI, L. GUERRIERO, M. MERLIN, Z. O'FRIEL, D. EVANS, F. HASSAN, K. K. NAGPAUL, M. SHAFI and A. WATAGHIN (Washington Meeting, May 1958) *Bull. Am. Phys. Soc.*, **3**, no. 3, p. 163.

(2) B. SECHI-ZORN and G. T. ZORN (priv. circulated preprint); E. HELMY, J. H. MULVEY, D. J. PROWSE and D. H. STORK (priv. circulated preprint); M. GRILLI, L. GUERRIERO, M. MERLIN, Z. O'FRIEL and G. A. SALANDIN (priv. circulated preprint and *Nuovo Cimento*, this issue).

(3) C. CEOLIN and L. TAFFARA: *Nuovo Cimento*, **5**, 435 (1957); D. AMATI and B. VITALE: *Nuovo Cimento*, **5**, 1533 (1957); H. P. STAPP: *Phys. Rev.*, **106**, 134 (1957); C. CEOLIN and L. TAFFARA: *Nuovo Cimento*, **6**, 425 (1957); C. CEOLIN and L. TAFFARA: *Reports of the Padua-Venice Conference* (September 1957).

is supposed to occur ([4]). Therefore it has been proposed some months ago ([5]) that the K^+-nucleon scattering could be alternatively interpreted according to the scheme of Fig. 2 in which a direct four boson $KK\pi\pi$-interaction has been supposed to exist.

In fact, the scanty experimental data that may be considered up to now as established, may be equally well interpreted according to the two schemes, with, in both cases, very reasonable values for the coupling constants G_{KNY} or $G_{KK\pi\pi}$.

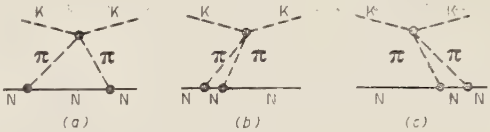

Fig. 2.

At first it appears that pion production should behave differently in the two cases. In fact, for direct K^+-nucleon interaction, we should interpret pion production as occurring according to the diagrams of Fig. 3 which are of the third order (while scattering is of second order).

On the contrary, in the case of $KK\pi\pi$ interaction, pion production should occur according to the diagram of Fig. 4 which are of the second order (while scattering in this case is of the third order).

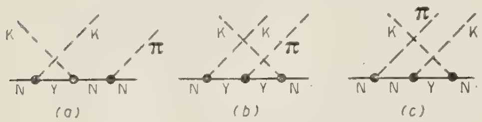

Fig. 3.

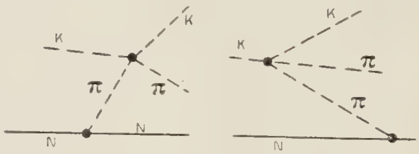

Fig. 4.

To give us a rough idea of the magnitude and the behaviour of the cross-section according to the two different assumptions a perturbative calculation should be sufficient at least at energies of the K-meson very close to the threshold.

In the present paper we give the results of the calculation we have made for the charge state reaction:

$$K^+ + p \to K^+ + \pi^+ + n\,.$$

As there are no definite experimental proofs for deciding if the K-Y-N interaction is scalar or pseudoscalar and as the ratio of pion production to scattering is probably rather insensitive to this assumption, we just choose the scalar case as the simpler.

([4]) B. T. FELD and N. COSTA: *Reports of the Padua-Venice Conference* (September 1958); M. KAWAGUCHI and M. J. MORAVCSIK: *Phys. Rev.*, **107**, 562 (1957); A. FUJII and R. E. MARSHAK: *Phys. Rev.*, **107**, 570 (1957).

([5]) Y. YAMAGUCHI: *Reports of the Padua-Venice Conference* (September 1958); S. BARSHAY: to be published in the *Phys. Rev.*

For scattering according to Fig. 1, the calculation gives:

$$\sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\mathrm{p}) = \frac{4\pi(g_\Lambda^2 + g_\Sigma^2)^2}{(M_\mathrm{Y} - M_\mathrm{N} + m_\mathrm{K} + T_i)^2}, \tag{1}$$

where the g_Y^2 are the rationalizated coupling constants ($g_\mathrm{Y}^2 = G_\mathrm{Y}^2/4\pi$), T_i represents the energy of the incident K-meson and we take $\hbar = c = 1$. The interaction constants can then be chosen in order to fit the magnitude of the cross section and to obtain that, for pure scattering, the $T = 1$ isotopic spin state should be dominant; this is achieved by taking:

$$g_\Lambda^2 \simeq 3\cdot g_\Sigma^2 = g^2\,, \qquad g_\Lambda^2 + g_\Sigma^2 = 1.34\,. \tag{2}$$

For the diagrams *a*), *b*) and *c*) of Fig. 3, we have naturally assumed for all the pion-baryon interactions the usual pseudovector coupling form with $f^2 = 0.08$. With some semplifying assumptions, the calculation is straightforward- and yields the result:

$$\text{3)}\quad \sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\pi^+\mathrm{n}) = \frac{8f^2\cdot g^4\cdot A^{\frac{3}{2}}\cdot m_\mathrm{K}^{\frac{1}{2}}\cdot m_\pi^{\frac{1}{2}}}{T_i^{\frac{1}{2}}(m_\mathrm{K} + T_i)}\cdot$$

$$\cdot\int\limits_0^{T_{\max}} \frac{T^{\frac{3}{2}}(Q - T\cdot B)^{\frac{1}{2}}[(\sqrt{3}-1)(m_\pi + T) - 2(M_\mathrm{Y} - M_\mathrm{N} + m_\mathrm{K} + T_i)]^2\,\mathrm{d}T}{(m_\pi + T)^3(m_\mathrm{K} - m_\pi + T_i - T)(M_\mathrm{Y} - M_\mathrm{N} + m_\mathrm{K} - m_\pi + T_i - T)^2(M_\mathrm{Y} - M_\mathrm{N} + m_\mathrm{K} + T_i)^2},$$

where the constants A, B, Q and $T_{\max}$ are given by:

$$A = \frac{2m_\mathrm{K}M_\mathrm{N}}{M_\mathrm{N} + m_\mathrm{K}}\,, \qquad B = 1 + \frac{m_\pi}{M_\mathrm{N} + m_\mathrm{K}}\,,$$

$$Q = T_i - m_\pi\,, \qquad T_{\max} = \frac{m_\mathrm{K} + M_\mathrm{N}}{m_\mathrm{K} + M_\mathrm{N} + m_\pi}\cdot Q$$

and the variable T is the kinetic energy of the π-meson in the final state.

The ratio of $\sigma(\mathrm{K}^+\mathrm{p} \to \mathrm{K}^+\pi^+\mathrm{n})$ to $\sigma(\mathrm{K}^+\mathrm{p} \to \mathrm{K}^+\mathrm{p})$ is, as one can easily see from (1) and (3) independent from the not well known coupling constant g^2 as it should be (+).

The calculation of the elastic scattering cross-section with the diagram of Fig. 2 can be easily obtained and we get:

$$\sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\mathrm{p}) = \frac{18\,g_{\mathrm{KK}\pi\pi}^2\cdot f^4}{(2\pi)^4\cdot m_\pi^4}\int\limits_0^\pi (H_1 + H_2)^2 \sin\alpha\,\mathrm{d}\alpha\,, \tag{4}$$

where

$$H_1 = \int\limits_0^\infty \frac{\boldsymbol{k}'^2\cdot\mathrm{d}\boldsymbol{k}'}{\omega'^2\cdot\omega''^2}\,, \qquad H_2 = \int\limits_0^\infty \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{k}')(\boldsymbol{\sigma}\cdot\boldsymbol{k}'')\,\mathrm{d}\boldsymbol{k}'}{\omega'^2\cdot\omega''(\omega' + \omega'')}\,, \tag{5}$$

(+) It is important to note that the form (3) of $\sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\pi^+\mathrm{n})$ is due to the particular hypothesis $g_\Lambda^2 = 3\cdot g_\Sigma^2$ *i.e.* in the general case, the ratio $\sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\pi^+\mathrm{n})/\sigma(\mathrm{K}^+\mathrm{p}\,|\,\mathrm{K}^+\mathrm{p})$ is independent from $g^2 = g_\Lambda^2 = \lambda\cdot g_\Sigma^2$ but not from λ.

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where ω' and ω'' are the total energies of the pions in the intermediate states, and α the scattering angle of K^+-meson.

The integration of the three variables in (5) has been performed for the angular part by developing the integrand functions in series of Legendre polynomials and then retaining only the terms with $l=0$ and $l=1$; and for the radial part graphically.

In order to fit the experimental data with the expression (4) we must take in this case:

$$g^2_{KK\pi\pi} = 2\,.$$

The total cross-section for pion production, according to the diagrams of Fig. 4, can be obtained in a similar way as in the case of direct interaction and we get:

$$\sigma(K^+p\,|\,K^+\pi^+n) = \frac{2\cdot A^{\frac{3}{2}}\cdot T_i^{\frac{1}{2}}\, m_K^{\frac{3}{2}} f^2\cdot g^2_{KK\pi\pi}}{\pi\cdot m_\pi^{\frac{1}{2}}(m_K+T_i)(m_\pi^2+2T_i m_K)^2}\int\limits_0^{T_{max}}\frac{T^{\frac{1}{2}}(Q-T\cdot B)^{\frac{1}{2}}\,dT}{(m_\pi+T)(m_K-m_\pi+T_i-T)}\,. \tag{6}$$

TABLE I.

Direct KYN *interaction.*

E_{kin} in MeV	$\sigma_{(elas.)}$ in mb	σ_π in mb $g_\Lambda^2=3g_\Sigma^2=g^2$	$(g_\Lambda^2+g_\Sigma^2)_{scal.}$	$\frac{\sigma_\pi}{\sigma_{(elas.)}}\,(g_\Lambda^2=3g_\Sigma^2)$
0	$9.9\cdot(g_\Lambda^2+g_\Sigma^2)^2$	—	1.34	—
100	$8.2\cdot(g_\Lambda^2+g_\Sigma^2)^2$	—	(This value has been obtained putting the total elastic scattering cross section at 100 MeV equal to 15 mb)	—
220	$6.8\cdot(g_\Lambda^2+{}_\Sigma^2 g)^2$	0		0
273	$6.2\cdot(g_\Lambda^2+{}_\Sigma^2 g)^2$	$3.62\cdot10^{-3}\cdot g^4$		$3.6\cdot10^{-5}$
334	$5.7\cdot(g_\Lambda^2+{}_\Sigma^2 g)^2$	$1.96\cdot10^{-2}\cdot g^4$		$2.1\cdot10^{-4}$

$KK\pi\pi$ *interaction.*

E_{kin} in MeV	$\sigma_{(elast.)}$ in mb	σ_π in mb	$g^2_{KK\pi\pi}$	$\frac{\sigma_\pi}{\sigma_{(elast.)}}$
0	$10.2\cdot g^2_{KK\pi\pi}$	—	2 (This value has been obtained putting the total elastic scattering cross-section at 100 MeV equal to 15 mb)	—
100	$7.5\cdot g^2_{KK\pi\pi}$	—		—
220	$6.2\cdot g^2_{KK\pi\pi}$	0		0
273	$5.72\cdot g^2_{KK\pi\pi}$	$5.54\cdot10^{-3}\, g^2_{KK\pi\pi}$		$0.97\cdot10^{-3}$
334	$5.15\cdot g^2_{KK\pi\pi}$	$1.5\cdot10^{-2}\, g^2_{KK\pi\pi}$		$2.03\cdot10^{-3}$

The ratio of pion production to elastic scattering also in this case is independent from the unknown $g^2_{KK\pi\pi}$ coupling constant.

The numerical data obtained in both cases are given in Table I. It is apparent that the ratio

$$\frac{\sigma(K^+p \mid K^+\pi^+n)}{\sigma(K^+p \mid K^+p)} \tag{7}$$

is markedly different near the threshold according to the two assumptions when the interaction constants are fitted in order to give the correct value of the total cross section for the elastic K$^+$-p scattering. In the second case it is about ten times larger than in the first one at 100 MeV energy above threshold. Moreover, the law of increase with the energy of the ratio (7) appears to follow different powers of the momentum in the two cases (Fig. 5).

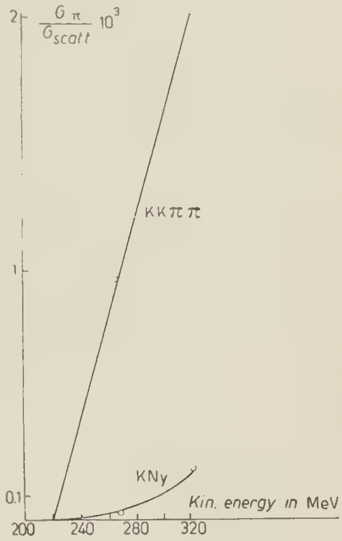

Fig. 5.

These results may give some hope that pion production in K$^+$ nucleon interactions could be used in order to obtain more information concerning the type of interaction between nucleon and K.

* * *

We are grateful to our colleagues of the plate group of this Institute for many helpful discussions on the experimental data concerning this problem.

Preliminary Results on K+ Interaction at High Energy (200 ÷ 350 MeV).

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(ricevuto il 16 Giugno 1958)

Technical details.

A stack of 240 Ilford G-5 pellicles ($20\,\text{cm}\times 17.5\,\text{cm}\times 14.5\,\text{cm}$) has been exposed to the doubly focussed K^+-beam ($p = 625$ MeV/c) at the Berkeley Bevatron and shared among the laboratories of Bristol, Dublin U.C. and Padova (×).

In order to pick up the tracks to be followed, the ionization of all tracks parallel to the beam within 5° has been measured. The values appear to be grouped in two peaks at $I/I_0 = 1.0 \pm 0.1$ (corresponding to the π^+) and $I/I_0 = 1.3 \pm 0.1$ (corresponding to the K^+).

Every K-track has been followed for ~ 9 cm, and the interactions have been recorded in the energy interval 200 ÷ ÷ 350 MeV. The nature and energy of the interacting particles have been determined by $g - \bar{\alpha}$ measurements. From these measurements and from $\Delta g/\Delta R$ measurements over a sample of tracks followed for ~ 20 cm, it was possible to determine the contamination of the selected tracks and the entrance energy of the K^+'s. The contamination was found to be negligible ($< 2\%$) and the entrance energy showed a flat distribution between 260 and 340 MeV.

Results.

Table I shows a summary of the events found on a total track length of 60.5 m.

We have accepted as K^+-H interactions all events satisfying the energy-momentum conservation; as an indication of their reliability we quote transversal momentum p_T normalized respectively to the primary momentum p_1, and to p_T standard deviation (σ) (evaluated for each event) (+).

(*) At present at the Bari University.

(+) Fulbright Research scholar on leave from St. Bonaventure University, Olean, N.Y., U.S.A. Z. O'Friel wishes to express his thanks to the United States Government for a Fulbright grant during the time of this research and gratefully acknowledge the kind hospitality of his colleagues for the use of their laboratories, especially Prof. N. Dallaporta and Prof. A. Rostagni, Director of the Institute of Physics, Padova.

(×) We are very grateful to Prof. J. E. Lofgren and his collaborators for the exposure of the stack. Our thanks are also due to the Bristol group for the processing of the plates and to the colleagues of Bristol, Dublin U.C., U.C.L.A. and B.N.L. who kindly supplied their data previous to publication.

(+) We have calculated the p_T distribution to be expected for K-bound proton collision (Fermi model; $p_{\max} = 241$ MeV/c). From this distribution and from the total number of two

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TABLE I.

Type	Scatterings ($\theta_{Lab} > 20°$)			Charge exchanges			Not analyzable	K-H	Decays in flight	Track followed (meters)
	a (*)	b (*)	Total	a (*)	b (*)	Total				
No. of events	16	52	68	1	28	29	8	7	8	60.5

(*) a = Events with no prongs (excluding the recoil),
b = Events with prongs (excluding the recoil).

The details on K^+-H events are as follows:

No. of event	E_1 (MeV)	θ_{CM}	p_T/p_1	p_T/σ
K_{IV} 678	230	76	0.026	0.47
62)	232	135	0.47	1.14
99	250	38	0.021	1.20
192	260	94	0.017	0.33
637	265	116	0.008	0.24
386	270	106	0.017	0.45
802	325	44	0.019	0.43

Table II shows the values of the mean free path for inelastic scattering (charge-exchange included).

The ratio charge exchanges/scattering turns out to be $.43 \pm .10$. In Fig. 1 are plotted the values of $\Delta E/E_1$ *vs.* θ_{Lab} for all scatterings. The curve reported in Fig. 1 corresponds to the K-free nucleon collisions. Fig. 2, 3 show the energy loss and the angular distributions, respectively. We report in Fig. 4 the prong distributions of the scattering (continuous line) and of charge exchanges (dotted line).

TABLE II.

Λ (emulsion) cm	Λ_1 (emulsion (*)) cm	λ (nuclear matter) Fermis
57 ± 6	55	6

(*) Λ_1 is the m.f.p. corrected for Coulomb effect.

prong scatterings, we have calculated that the contamination in our K-H sample is less than 0.5 event. Further details and results will be given at the Geneva Conference.

We have found a one-prong star, in wich a positive pion has been produced by collision of a K with a nucleus. The nature and the energy of the primary K has been determined by accurate measurements of $\bar{\alpha}-g$. The nature of the prong comes out from the typical $\pi^+\to\mu^+\to e^+$ decay at rest. The energy of the π has been determined by its range.

The main data are:

Particle	Energy (MeV)	θ_{Lab}	Remarks
K	300 ± 24	—	—
π	46	89	π-μ-e at rest $R=30.6$ mm

We can exclude this event to be a decay in flight of a K. In fact, if we should interpret this event as a decay in flight, the energy of the π in center of mass system should be 165 ± 15 MeV (*i.e.* clearly greater than the energy of a π from a $K_{\pi2}$ decay).

The elementary reaction:

$$K^+ + n = K^0 + n + \pi^+$$

is energetically possible.

Discussion.

1) Our data on K-H collisions seem to indicates an increased σ_{Kp} with respect to the results at lower energy ($\sigma=(14.5\pm2.2)$ mb for $E_1<220$ MeV). By adding the data of other laboratories (1) we obtain $\sigma_{Kp}=(24^{+8}_{-7})$ mb (E_1 between 200 and 350 MeV).

2) The quoted value of $\sigma_{\text{K-nucleus}}$ for inelastic events is in agreement with that found by other authors (1). From all these values a clear trend appears of the $\sigma_{\text{K-nucleus}}$ to increase also for $E_1>200$ MeV, as for the lower energy (2,3).

The value of the m.f.p. (λ) in nuclear matter at $E_1\sim125$ MeV enables us to evaluate the probability of double scatterings of a K^+ inside one nucleus. This turns out to be $\sim32\%$ (assuming $E_1=250$ MeV and $\Delta E/E_1=50\%$ in the first collision). Due to such an high frequency of double scatterings the interpretation of $(d\sigma/d\Omega)_{Lab}$ (Fig. 3), *i.e.* the valuation of the $(d\sigma/d\Omega)_{CM}$ for K-nucleon collision, becomes rather complicated. Some events with large $\Delta E/E_1$ are inconsistent with a single collision K-nucleon within the nucleus (Fig. 1). These events are attributable to double scatterings of K^+ within the same nucleus. It seems unlikely that such events are due to pion production and reabsorption, owing to the small cross-section of the pion production reaction.

3) In Fig. 5 we report a summary of the present data on the ratio charge exchanges scattering. The relative frequency of charge exchanges to the inelastic events for $E_1>200$ MeV is clearly greater than at lower energies. To obtain the true (*i.e.* in the interaction K-free nucleon) variation of the ratio C.E./scattering with E_1, corrections should be applied to the experimental values to account for the effect of Pauli principle and of double scatterings. We point out that the occurrence of double scatterings at high energy leads to an increase of the experimental relative frequency of charge exchange to scattering events. At $E_1=(200\div350)$ MeV ($\sim32\%$ of double scatterings) a true ratio C.E./scatt.$=0.20$ (pure $T=1$ state) would give an experimental ratio 0.27.

(1) Communications of the plate groups of Bristol, Los Angeles (U.C.L.A.), and Padua laboratories at Washington meeting (May 1958), reported by Dr. D. J. PROWSE.

(2) See Fig. 1 in B. SECHI-ZORN and G. T. ZORN: *Phys. Rev.*, **108**, 1098 (1957).

(3) See Fig. 4 in M. GRILLI, L. GUERRIERO, M. MERLIN and G. A. SALANDIN: *Analysis of inelastic interactions of K+-mesons with emulsion nuclei* (40 ÷ 150 MeV) (circulated preprint, submitted to *Nuovo Cimento*).

The correction to this ratio at these energies arising from the Pauli principle is less than 0.03. The observed value C.E./scatt. = 0.43 (*) shows therefore a clear contribution of a $T = 0$ state.

In a previous analysis of inelastic scattering of K$^+$ we found the presence of $T = 0$ state also for $E_1 = (80 \div 150)$ MeV, [3] with a true ratio C.E./scatt. = 0.28 ± 0.10. The quoted value 0.43 seems to indicate that the contribution of the $T = 0$ state is more rapidly increasing with the energy than the contribution of the $T = 1$ state.

These results could be explained assuming that the K-nucleon potential is attractive in the $T = 0$ state, and repulsive in the $T = 1$ state, in agreement with the signs of the scattering amplitudes evaluated by us [3] for $E_1 = (80 \div 150)$ MeV ($a_{10} < 0$; $a_{01} > 0$). If it is so, the $T = 0$ state should give a resonance at higher energies.

4) Further support for the reliability of the elementary reactions:

(I) $K^+ + p = K^+ + p$,

(II) $K^+ + n = K^+ + n$,

(III) $K^+ + n = K^0 + p$,

may be obtained from the following considerations.

a) In five charge exchange events the dynamics of the collision (angle and energy of emitted proton) is consistent, within the errors, with a K-n collision (the neutron being pratically at rest).

b) The average number of prongs is greater for charge exchange events ($n = 2.6$), than for scatterings ($n = 1.6$), (see Fig. 4) as it is to be expected on the basis of the reactions (I), (II) and (III).

(*) The value reported at Washinghton Meeting [1] for this ratio is 0.42 ± 0.05.

Starting from these values of n it is also possible to evaluate roughly the frequency of reactions (I) and (II). We found that the contribution of the reaction (II) is certainly at least comparable with that of the reaction (I) at $E_1 > 200$ MeV.

5) Until now three cases [4,5] have been reported of charged π produced in K$^+$-nucleus collisions, on a total of ~ 500 inelastic events. To evaluate the frequency of π production in K-nucleon collisions we should take in to account:

a) Possible π^0-production. This reaction would give rise to events with a very large $\Delta E/E_1$. In fact, such events have been observed (see Fig. 1). In our cases, however, the emission of a π^0 is to be excluded by energy considerations.

b) Production and subsequent absorption of a π in the same nucleus, charge exchanges of charged π. The evaluation of these effects is now in progress.

Neglecting these corrections, the relative frequency

$$\frac{\pi\text{-production}}{\text{Inelastic events}}$$

can be roughly evaluated and results to be ~ 0.006. This value is larger than predicted by preliminary theoretical calculations, assuming either a direct π production by nucleon via a KNY interaction or a K K$\pi\pi$ interaction (*).

[4] B. Sechi-Zorn and G. T. Zorn: *The production of a π^--Meson by a* K$^+$*-particle* (circulated preprint, October 1957).

[5] E. Helmy, J. H. Mulvey, D. J. Prowse and D. M. Stork: *An example of the production of a π^--Meson by a* K$^+$*-Meson* (circulated preprint, April 1958).

(*) C. Ceolin, N. Dallaporta and L. Taffara: *Pion production in the* K$^+$*-nucleon interaction* (*Nuovo Cimento*, in the same issue). We are very grateful to these colleagues for many illuminating discussions on this subject.

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On the Stability of the Nordström Reissner Singularity (*).

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It has been shown by Regge and Wheeler [1] that a Schwarzschild singularity endowed with mass will undergo small displacements about the spherical form and will therefore remain stable if subjected to a small non spherical perturbation.

We wish to consider here the more general problem of the stability of the Nordström Reissner distribution of mass and electrical charge given by the metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = e^\varrho\,\mathrm{d}t^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) - e^{-\varrho}\,\mathrm{d}r^2\,; \qquad \Phi_0 = \frac{q}{r}\,,$$

where

$$e^\varrho = 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,; \quad m = \frac{M\gamma}{c^2}\,; \quad q = \frac{Q\gamma^{\frac{1}{2}}}{c^2}\,; \qquad q < m\,,$$

and γ is the gravitational constant, M the mass and Q the charge.

We wish to prove also in this case the stability of the metric against small perturbations.

Replacing in the well known equations of Einstein Maxwell the metric tensor $g_{\mu\nu}$ by $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ and the vector potential Φ_μ by $\bar{\Phi}_\mu = \Phi_\mu + \varphi_\mu$ and neglecting non linear terms in $h_{\mu\nu}$, φ_μ and their derivatives one obtains the following new system of equations:

$$(1) \qquad \left\{\begin{aligned} &\delta R_{\mu\nu} = \frac{8\pi\gamma}{c^4}\,\delta T_{\mu\nu} \\ &\delta F_{\mu\ ,\nu}^{\ \nu} = 0 \end{aligned}\right.\,,$$

where $R_{\mu\nu}$ is Ricci's tensor $T_{\mu\nu}$ the energy momentum tensor $F_{\mu\nu}$ is the electromagnetic field tensor which is connected to the covariant derivative of Φ_μ, $\Phi_{\mu,\nu}$ by the relation

$$F_{\mu\nu} = \Phi_{\mu,\nu} - \Phi_{\nu,\mu}\,.$$

We may expand $h_{\mu\nu}$ and φ_μ in terms of tensor spherical harmonics. These can

(*) This is a brief account of a thesis in theoretical Physics (Torino, Dec. 1957).

(1) T. Regge and J. A. Wheeler: *Phys. Rev.*, **108**, 1063 (1957).

be grouped in two classes according to their opposite parity

odd waves

$$h_{\mu\nu} = \left\|\begin{matrix} 0 & 0 & 0 & r^2 y_0 \\ 0 & 0 & 0 & e^{-\varrho} y_1 \\ 0 & 0 & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{matrix}\right\| \sin\theta \frac{dY_J^0}{d\theta} e^{\omega\tau}; \qquad \varphi_\mu = \left\|\begin{matrix} 0 \\ 0 \\ 0 \\ i\Phi \end{matrix}\right\| \sin\theta \frac{dY_J^0}{d\theta} e^{\omega\tau},$$

even waves

$$h^u{}_v = \left\|\begin{matrix} -N & -Me^{-\varrho} & 0 & 0 \\ -Me^{\varrho} & N & 0 & 0 \\ 0 & 0 & -K & 0 \\ 0 & 0 & 0 & -K \end{matrix}\right\| Y_J^0 e^{\omega\tau}; \qquad \varphi_\mu = \left\|\begin{matrix} -iW \\ -iG \\ 0 \\ 0 \end{matrix}\right\| Y_J^0 e^{\omega\tau},$$

where we have put $\tau = it = x_0$; $x_1 = r$, $x_2 = \theta$, $x_3 = \varphi$.

From equations (1) with $\gamma = 1$, $c = 1$ one deduces then for the radial dependence of our tensors the following systems

odd waves

$$(2)\qquad \begin{cases} y_1' + \omega r^2 e^{-\varrho} y_0 = 0\,, \\ \omega y_0' + y_1 \dfrac{e^{-\varrho}}{r^2}\left\{\dfrac{(J+2)(J-1)}{r^2} e^{\varrho} - \omega^2\right\} - \dfrac{4\omega q}{r^4}\,\Phi = 0\,, \\ \psi' + \left\{\omega^2 e^{-\varrho} - \dfrac{J(J+1)}{r^2} + \dfrac{4q^2}{r^4}\right\}\Phi - \dfrac{1}{\omega} q \dfrac{(J+2)(J-1)}{r^4} y_1 = 0\,, \\ \Phi' - e^{-\varrho}\psi = 0\,, \end{cases}$$

even waves

$$(3)\qquad \begin{cases} a) \quad -K' - N' - \varrho' N + \omega M e^{-\varrho} = -\dfrac{4q}{r^2} e^{-\varrho} W\,, \\ b) \quad (Me^{\varrho})' + \omega(N-K) = \dfrac{4q}{r^2} e^{\varrho} G\,, \\ c) \quad -K' - \dfrac{N+K}{r} + \dfrac{J(J+1)}{2\omega r^2} M + \dfrac{\varrho'}{2} K = 0\,, \\ d) \quad (e^{\varrho} G)' = -\omega W e^{-\varrho}\,, \\ e) \quad e^{-\varrho}(W' - \omega G) + \dfrac{J(J+1)}{r^2} G + \dfrac{q}{r^2} e^{-\varrho} K = 0\,, \\ f) \quad N\left[\dfrac{4q^2}{r^2} - \dfrac{6m}{r} - (J+2)(J-1)\right] + M\left[\dfrac{J(J+1)}{2\omega}\varrho' e^{\varrho} - 2\omega r\right] + \dfrac{8q}{r} W - \\ \quad - \dfrac{4q}{r^2}\dfrac{J(J+1)}{\omega} e^{\varrho} G + K\left[2\omega^2 r^2 e^{-\varrho} - (J+2)(J-1) + \dfrac{r^2 \varrho'^2 e^{\varrho}}{2} - \dfrac{2m}{r} - \dfrac{2q^2}{r^2}\right] = 0\,. \end{cases}$$

The solutions show fuchsian singularities in $r=0$, $r=\alpha_1, \alpha_2$ $(\alpha_1 < \alpha_2)$ which are the zeros of e^ϱ. There is also an essential singularity at infinity. We can exclude the points 0 and α_1 since they belong to the unphysical region. Stability implies that for complex ω there is no bound solution between α_2 and infinity. Let us consider the case of odd waves, and purely imaginary ω. It is easy to deduce from eqs. (2) if one eliminates the variables y_0 and ψ the following identity:

$$(4)\qquad \int_{\alpha_2}^{\infty}\left[g_1(r)\,|y_1|^2 + q^2\frac{e^\varrho}{r^2}\,|y_1'|^2 + g_2(r)\,|\Phi|^2 + \nu^2 e^\varrho\,|\Phi'|^2\right]\mathrm{d}r = \left[\frac{e^\varrho y_1' y_1^*}{r^2}\right]_{\alpha_2}^{\infty} + [\nu^2 e^\varrho \Phi^{*\prime}\Phi]_{\alpha_2}^{\infty}\,,$$

with

$$g_1(r) = q^2\left[\frac{(J+2)(J-1)}{r^4} + \frac{\nu^2 e^{-\varrho}}{r^2}\right];\quad g_2(r) = \nu^2\left[\frac{J(J+1)}{r^2} + \nu^2 e^{-\varrho} - \frac{4q^2}{r^4}\right],\quad \nu = i\omega\,.$$

In eq. (4) we have a definite positive quadratic form when $q/m \leqslant 0.94$ for $J=1$ and $q/m < 1$ for $J \geqslant 2$.

For all possible solutions finite at α_2 and ∞ we can now assert that the integrals on the right hand side of eq. (4) must vanish for ω imaginary. Therefore these solutions do not exist.

For even waves one could in principle carry out a similar analysis; there is however the disturbing complication of a system which has six equations in five unknowns. By a suitable choice of initial conditions for N, M, K, W, G, the last equation of system (3) is compatible with the preceding ones.

This equation serves for eliminating a spurious solution of (3; $a, b, \ldots e$) which has no physical meaning. We did not succeed in proving the stability in the even case in the same rigorous manner used for the odd waves. One can however produce qualitative arguments in favour of stability as were already presented for the Schwarzschild metric [1].

* * *

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